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DERIVATION OF AN AUTOMATIC AIRCRAFT ELEVATOR CONTROLLER

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DERIVATION OF AN AUTOMATIC

AIRCRAFT ELEVATOR CONTROLLER

by

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[redacted]

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ABSTRACT

An automatic elevator controller for the final phase of an Instrument Landing System approach is designed using optimization theory and the practicality of the controller investigated. The problem is discussed and the assumptions stated. Then a mathematical model for the aircraft and a desired flare-out approach path are derived. The aircraft and approach limitations are established and the model is tested. Dynamic Programming and the Parametric Expansion Method provide the optimal control from which the controller is designed. A computer program is developed to investigate the controller. The results are inconclusive and a recommendation for further study is made.

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TABLE OF SYMBOLS AND ABBREVIATIONS

T (Superscript)	=	matrix transpose operator
. (Superscript)	=	first time derivative
.. (Superscript)	=	second time derivative
... (Superscript)	=	third time derivative
IV (Superscript)	=	fourth time derivative
d (Subscript)	=	desired value of the variable
e (Subscript)	=	denotes an equilibrium condition
s (Subscript)	=	denotes an aircraft constant
A	=	matrix relating $\dot{\underline{z}}(t)$ and $\underline{z}(t)$
a_{ij}	=	an element of A matrix
a_i	=	constants of the desired approach
a	=	an arbitrary instant of time
B	=	matrix relating $\dot{\underline{z}}(t)$ and u(t)
b_i	=	an element of B matrix
D	=	an arbitrary distance from the end of the runway
E	=	minimum value of the error index, J. called the minimum error function
F	=	matrix of weighting factors for system errors at the final time
f_{ij}	=	an element of F matrix
f(a)	=	an arbitrary function of time evaluated at the instant a
h(t)	=	aircraft altitude
ILS	=	Instrument Landing System
J	=	system error index
K_s	=	aircraft short period gain

$k_{mp}(t)$ = time functions of E

m = summation index

p = summation index

P_{ij} = time variant coefficients of the system signal $x_i(t)x_j(t)$

$Q(t)$ = time variant matrix of weighting factors for the system errors

q_{ij} = an element of $Q(t)$ matrix

s = complex variable

T_s = aircraft path time constant

t = real time

t_1 = time aircraft is over end of runway on the desired approach

t_f = desired touchdown time

$u(t)$ = elevator control signal

V = aircraft velocity

W = aircraft weight

W_s = aircraft short period resonant frequency

X = aircraft x axis

$\underline{x}(t)$ = measured signal vector

$x_i(t)$ = element of $\underline{x}(t)$ vector

Z = aircraft z axis

$\underline{z}(t)$ = vector of changes in the measured signals

$z_i(t)$ = element of $\underline{z}(t)$ vector

$\alpha(t)$ = aircraft angle of attack

α_s = angle of attack when aircraft stalls

Δ	=	denotes changes in the variable about some equilibrium condition
$\delta(t)$	=	elevator deflection
$\gamma(t)$	=	glide path angle
$\theta(t)$	=	pitch angle
ξ	=	aircraft short period damping factor

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1. INTRODUCTION

This study is an attempt to synthesize an automatic control mechanism for the terminal phase of an aircraft landing using optimization theory. The problem was originally presented by F. J. Ellert and C. W. Merrian [2]. Dynamic Programming [1] and the Parametric Expansion Method [2] are used to design a simple controller.

Section two describes the problem and presents the assumptions made to simplify the initial study. Originally the investigation of Ellert and Merriam was to be continued but after several weeks of investigation, it was found that their equations for the aircraft represented only the changes about an equilibrium condition [3] rather than the true signals. Equations representing only the changes are derived in Appendix A and are used in section three to obtain different equations for the actual aircraft signals.

In section four a more realistic desired approach is defined and equations for the desired values of the measured signals are derived. The performance limits and system constraints are discussed in section five. Section six covers the testing of the aircraft model.

Section seven develops the theory to obtain the optimal control in terms of the measured signals, aircraft constants, and functions of time. The necessary first-order, linear differential equation to obtain the time functions for the optimal control are derived in Appendix B. Section eight outlines the mechanization of a controller using the optimal control equation. Section nine discusses the computer programming and the investigation conducted with the results in section ten and the conclusions in section eleven.

2. PROBLEM STATEMENT

Landing an aircraft in inclement weather when the pilot has no visual contact with the runway until the last few hundred feet is a function performed at many airports by an Instrument Landing System, commonly referred to as ILS.

The aircraft follows an electronic beam which is set to a glide path of approximately three degrees and provides an azimuth reference to the runway. As the aircraft descends, the beam resolution becomes poor and sizable altitude errors can occur. The three degree glide path is normally followed to a point within half a mile from the approach end of the runway. The pilot then takes over visually to land the aircraft or is waved off to try again.

At this time, a choice must be made by the pilot to follow ILS and hold the rate of descent he has or to make a flare-out approach to touch down on the runway with a lesser rate of descent. The flare-out approach puts less stress on the aircraft and is a more comfortable landing.

A controller is needed to accomplish the flare-out approach so that if visibility is restricted, this kind of landing may still be made. The following assumptions are made to simplify these first studies:

a. The aircraft is laterally aligned with the runway. Only errors in the vertical plane will be considered.

b. At one-half mile from the end of the runway, the aircraft will be waved off if altitude, rate of descent, pitch angle, and pitch rate exceed specified limits.

c. Wind effects will not be considered.

3. SYSTEM MODEL DERIVATION

The first step in the synthesis of an aircraft control system is the development of a mathematical model relating the control (in this case, elevator deflection) to some measurable response variables. It is essential that any model chosen be tested to insure that it follows all known responses of the real system. This will be discussed further in section six.

Figure 1 defines the aircraft coordinates, angles, and positive direction of elevator deflection. Equations (3.1) and (3.2) have been derived to describe the motion of an aircraft [2] under the following restrictive conditions:

a. The equations are linearized with the assumption that deviations from an equilibrium flight path are small. Only the changes about this equilibrium condition are represented by equations (3.1) and (3.2).

b. The glide path angle, γ, is small enough that the small angle approximations, $\sin\gamma \approx \gamma$ and $\cos\gamma = 1$ can be used. This approximation is valid for the landing paths to be investigated.

c. It is assumed that the aircraft velocity, V, is held constant.

Note that should a controller be found that is feasible from this basic synthesis, a more realistic model of the aircraft should be used for further studies.

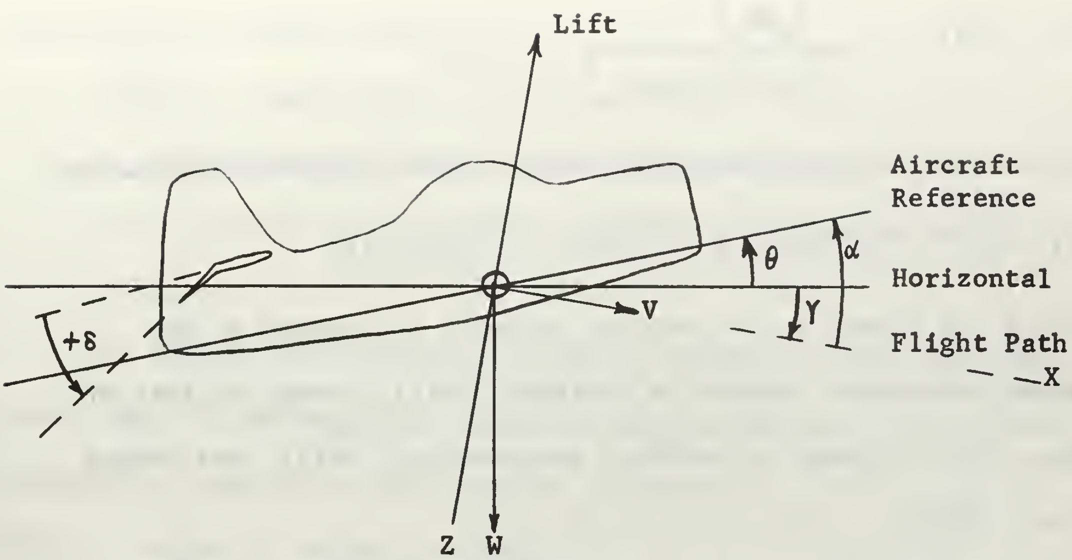


Figure 1. Definition of Aircraft Coordinates, Angles, and Elevator Deflection.

$$(3.1) \quad \Delta\dot{\theta}(s) = \frac{K_s W_s^{\,2}(T_s s + 1)}{s^2 + 2\xi W_s s + W_s^{\,2}} \Delta\delta(s)$$

$$(3.2) \quad \Delta\ddot{h}(s) = \frac{V}{(T_s s + 1)} \Delta\dot{\theta}(s)$$

where $\Delta\dot{\theta}(s)$ = Laplace transform of the change in pitch rate, $\Delta\dot{\theta}(t)$.

$\Delta\delta(s)$ = Laplace transform of the change in elevator deflection, $\Delta\delta(t)$.

$\Delta\ddot{h}(s)$ = Laplace transform of the change in vertical acceleration, $\Delta\ddot{h}(t)$.

K_s, T_s, W_s, ξ are constants with different values dependent on the type of aircraft.

Equations (3.1) and (3.2) can be combined to provide a transfer function relating the change in vertical acceleration and elevator deflection,

$$(3.3) \quad \Delta\ddot{h}(s) = \frac{K_s V W_s^{2}}{s^2 + 2\xi W_s s + W_s^{2}} \Delta\delta(s)$$

which can be written as a fourth order linear differential equation

$$(3.4) \quad \Delta\overset{IV}{h}(t) + 2\xi W_s \Delta\dddot{h}(t) + W_s^{2}\Delta\ddot{h}(t) = K_s V W_s \Delta\delta(t)$$

relating the change in the control variable to changes in the response variables: change in altitude, $\Delta h(t)$; change in rate of ascent, $\Delta\dot{h}(t)$; change in vertical acceleration, $\Delta\ddot{h}(t)$; and change in jerk, $\Delta\dddot{h}(t)$.

All of these signals are not directly measurable. Signals that are readily measurable in the aircraft are desired. These are:

a. Altitude, $h(t)$, which can be measured from a barometric altimeter or radar altimeter.

b. Rate of ascent, $\dot{h}(t)$, measured with a barometric rate meter.

c. Pitch angle, $\theta(t)$.

d. Pitch rate, $\dot{\theta}(t)$, both measurable from gyros.

These are expressed in (3.5) as an equilibrium condition plus a change about that signal.

$$(3.5) \quad \begin{aligned} x_1(t) &= h_e(t) + \Delta h(t) \\ x_2(t) &= \dot{h}_e(t) + \Delta\dot{h}(t) \\ x_3(t) &= \theta_e(t) + \Delta\theta(t) \\ x_4(t) &= \dot{\theta}_e(t) + \Delta\dot{\theta}(t) \end{aligned}$$

where the subscript e denotes the equilibrium condition.

A set of first-order differential equations can be developed from (3.2) and (3.4) for the changes about equilibrium and result in equations of the form

$$(3.6) \quad \dot{\underline{z}}(t) = A\underline{z}(t) + Bu(t)$$

where $\underline{z}(t)$ is a 4x1 vector and

$$\underline{z}^T(t) = [\Delta h(t), \Delta \dot{h}(t), \Delta\theta(t), \Delta\dot{\theta}(t)]$$

A is a known 4x4 matrix of aircraft constants

B is a known 4x1 distribution vector of the scalar control, $u(t)$.

The derivation of (3.6) is given in appendix A and represents the change of the measurable signals, $\underline{x}(t)$, about the equilibrium condition. Equations (3.5) can now be written

$$(3.7) \quad \begin{aligned} x_1(t) &= h_e(t) + z_1(t) \\ x_2(t) &= \dot{h}_e(t) + z_2(t) \\ x_3(t) &= \theta_e(t) + z_3(t) \\ x_4(t) &= \dot{\theta}_e(t) + z_4(t) \end{aligned}$$

The equilibrium condition, $\underline{x}_e(t)$, is chosen to be a constant rate of descent, as variations about an ideal ILS approach are to be investigated. $\underline{x}_e(t)$ becomes

$$(3.8) \quad \begin{aligned} h_e(t) &= h(0) + \dot{h}(0)t \\ \dot{h}_e(t) &= \dot{h}(0) \\ \theta_e(t) &= \theta(0) \\ \dot{\theta}_e(t) &= 0 \end{aligned}$$

Then (3.7) can be expressed as

$$(3.9) \quad \begin{aligned} x_1(t) &= h(0) + \dot{h}(0)t + z_2(t) &&= x_1(0) + x_2(0)t + z_1(t) \\ x_2(t) &= \dot{h}(0) + z_2(t) &&= x_2(0) + z_2(t) \\ x_3(t) &= \theta(0) + z_3(t) &&= x_3(0) + z_3(t) \\ x_4(t) &= z_4(t) \end{aligned}$$

where $z_1(0) = z_2(0) = z_3(0) = 0$ and $z_4(0) = x_4(0)$.

A set of linear differential equations describing $\underline{x}(t)$ is

$$
\begin{aligned}
(3.10) \quad \dot{x}_1(t) &= \dot{h}(0) + \dot{z}_1(t) \\
\dot{x}_2(t) &= \dot{z}_2(t) \\
\dot{x}_3(t) &= \dot{z}_3(t) \\
\dot{x}_4(t) &= \dot{z}_4(t)
\end{aligned}
$$

substituting (A.9) in (3.10) gives

$$
\begin{aligned}
(3.11) \quad \dot{x}_1(t) &= \dot{h}(0) + z_2(t) \\
\dot{x}_2(t) &= a_{22}z_2(t) + a_{23}z_3(t) \\
\dot{x}_3(t) &= z_4(t) \\
\dot{x}_4(t) &= a_{42}z_2(t) + a_{43}z_3(t) + a_{44}z_4(t) + b_4u(t)
\end{aligned}
$$

using (3.9), (3.11) can be written

$$
\begin{aligned}
(3.12) \quad \dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= a_{22}[x_2(t) - x_2(0)] + a_{23}[x_3(t) - x_3(0)] \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= a_{42}[x_2(t) - x_2(0)] + a_{43}[x_3(t) - x_3(0)] \\
&\quad + a_{44}x_4(t) + b_4u(t)
\end{aligned}
$$

which are the desired linear equations as a function of the signals, $\underline{x}(t)$, and the control, $u(t)$. The system model is also dependent on the equilibrium values of the descent rate and pitch angle because of the chosen equilibrium condition.

4. DESIRED FLARE-OUT APPROACH

At time $t = 0$ with the aircraft some horizontal distance, D, from the end of the runway, it is desired to have the aircraft leave the three degree glideslope, increase its rate of descent, then flare-out to arrive over the end of the runway at time, $t = t_1$, at a lower altitude than if it had remained on the glideslope beam and with a lower rate of descent. From time, $t = t_1$, until time of touchdown, $t = t_f$, the rate of descent is held constant and all accelerations are zero. Figure 2 shows this desired path. An exponential altitude path over the time interval, $t_1 \leq t \leq t_f$, would be more realistic but the constant rate of descent was chosen for simplicity.

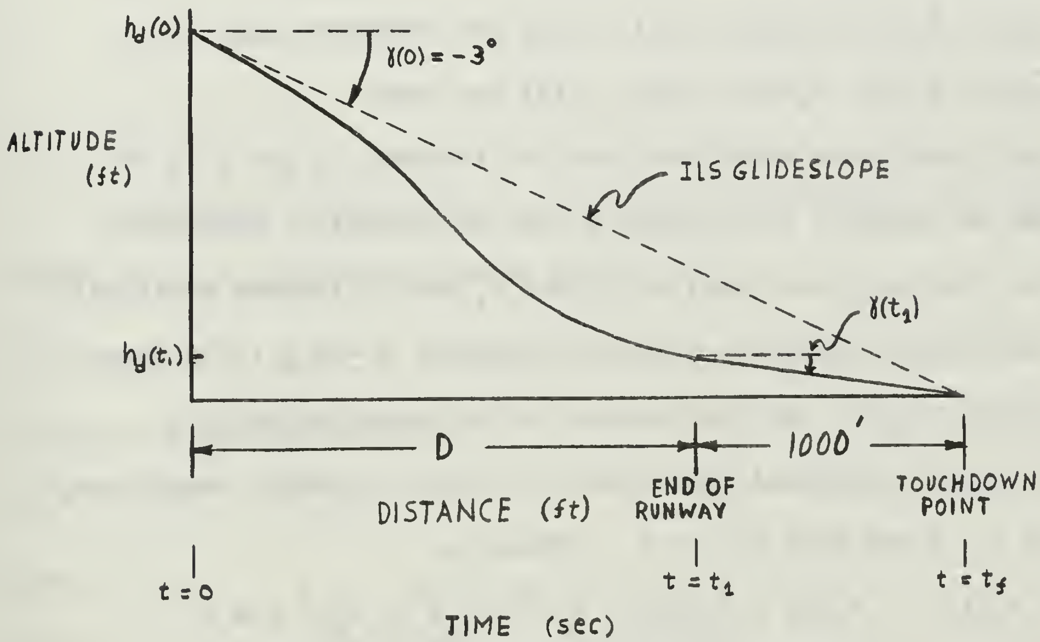


Figure 2. Desired Flare-out Approach

Note: All desired approach variables will be subscripted, d, to differentiate from the actual aircraft model signals.

The following assumptions are made:

a. Origin of the ILS glideslope beam and the desired touchdown point coincide 1000 feet down the runway.

b. Prior to $t = 0$, the aircraft has been following an ideal ILS approach.

From the problem definition and the general desired approach specifications, the known conditions must be compiled to derive equations for the desired measurable signals.

First the known conditions at times $t = 0$, $t = t_1$, and $t = t_f$ will be stated. Using these boundary conditions, equations will then be derived for the time intervals $0 \leq t < t_1$ and $t_1 \leq t \leq t_f$.

Prior to time $t = 0$, the aircraft is descending on an ideal ILS approach. Since all measurable signals are continuous, the following conditions exist at $t = 0$:

$h_d(0)$, $\dot{h}_d(0)$, $\theta_d(0)$, $\alpha_d(0)$, $\gamma_d(0)$ are constants and

$\ddot{h}_d(0)$, $\dddot{h}_d(0)$, $\dot{\theta}_d(0)$, $\dot{\alpha}_d(0)$, $\dot{\gamma}_d(0)$ are zero.

The flight path specified over the interval $t_1 \leq t \leq t_f$ is the same as before $t = 0$ (constant rate of descent). Continuity requires the same conditions at t_1 and t_f with different constants.

The desired flare-out approach path over $0 \leq t \leq t_1$ in terms of altitude, $h_d(t)$, and derivatives can be approximated by a seventh order polynomial since there are eight boundary conditions, four at $t = 0$ and four at $t = t_1$. These are

$$
\begin{aligned}
(4.1) \quad h_d(t) &= h_d(0) + \dot{h}_d(0)t + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 \\
\dot{h}_d(t) &= \dot{h}_d(0) + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + 7a_7t^6 \\
\ddot{h}_d(t) &= 12a_4t^2 + 20a_5t^3 + 30a_6t^4 + 42a_7t^5 \\
\dddot{h}_d(t) &= 24a_4t + 60a_5t^2 + 120a_6t^3 + 210a_7t^4
\end{aligned}
$$

where $a_4 = [20\gamma(0) + 15\gamma(t_1)] \dfrac{V}{(t_1)^3} - \dfrac{35}{(t_1)^4}[h_d(0) - h_d(t_1)]$

$$a_5 = [45\gamma(0) + 39\gamma(t_1)] \frac{-V}{(t_1)^4} + \frac{84}{(t_1)^5}[h_d(0) - h_d(t_1)]$$

$$a_6 = [36\gamma(0) + 34\gamma(t_1)] \frac{V}{(t_1)^5} - \frac{70}{(t_1)^6}[h_d(0) - h_d(t_1)]$$

$$a_7 = [10\gamma(0) + 10\gamma(t_1)] \frac{-V}{(t_1)^6} + \frac{20}{(t_1)^7}[h_d(0) - h_d(t_1)]$$

For the interval $t_1 \leq t \leq t_f$, the desired flight path is a constant rate of descent. The desired equations are:

$$(4.2)\quad h_d(t) = h_d(t_1) + \dot{h}_d(t_1)[t-t_1]$$

$$\dot{h}_d(t) = \dot{h}_d(t_1)$$

$$\ddot{h}_d(t) = 0$$

$$\dddot{h}_d(t) = 0$$

The desired pitch angle, $\theta_d(t)$, can be expressed as its equilibrium value, $\theta_d(0)$, plus changes about $\theta_d(0)$:

$$(4.3)\quad \theta_d(t) = \theta_d(0) + \Delta\theta_d(t).$$

Taking the derivative gives

$$(4.4)\quad \dot{\theta}_d(t) = \Delta\dot{\theta}_d(t).$$

From (3.2),

$$(4.5)\quad \Delta\dot{\theta}_d(t) = \frac{Ts}{V} \Delta\dddot{h}_d(t) + \frac{1}{V}\Delta\ddot{h}_d(t).$$

Integrating (4.5)

$$(4.6)\quad \Delta\theta_d(t) = \frac{Ts}{V}\Delta\ddot{h}_d(t) + \frac{1}{V}\Delta\dot{h}_d(t).$$

The desired values of (3.5) can now be written in terms of the derived equations

(4.7)

$$x_{1d}(t) = \begin{cases} h_d(0) + \dot{h}_d(0)t + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7, \\ 0 \leq t \leq t_1 \\ h_d(t_1) + \dot{h}_d(t_1)[t-t_1], \; t_1 \leq t \leq t_f \end{cases}$$

$$x_{2d}(t) = \begin{cases} \dot{h}_d(0) + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + 7a_7t^6, \\ 0 \leq t \leq t_1 \\ \dot{h}_d(t_1), \; t_1 \leq t \leq t_f \end{cases}$$

$$x_{3d}(t) = \begin{cases} \theta_d(0) + \dfrac{T_s}{V}(12a_4t^2 + 20a_5t^3 + 30a_6t^4 + 42a_7t^5) \\ + \dfrac{1}{V}(4a_4t^3 + 5a_5t^4 + 6a_6t^5 + 7a_7t^6), \; 0 \leq t \leq t_1 \\ \theta_d(t_1), \; t_1 \leq t \leq t_f \end{cases}$$

$$x_{4d}(t) = \begin{cases} \dfrac{T_s}{V}(24a_4t + 60a_5t^2 + 120a_6t^3 + 210a_7t^4) \\ + \dfrac{1}{V}(12a_4t^2 + 20a_5t^3 + 30a_6t^4 + 42a_7t^5), \\ 0 \leq t \leq t_1 \\ 0, \; t_1 \leq t \leq t_f \end{cases}$$

5. PERFORMANCE LIMITS AND SYSTEM CONSTRAINTS

Aircraft constants, numerical values of the desired flight path, and variable limits must be specified before testing the system model. The T-28 aircraft was chosen for this investigation. Values of the aircraft constants in (3.1) and (3.2) for the T-28 in a landing configuration are:

$$(5.1) \qquad \begin{aligned} K_s &= -1.350 \text{ sec}^{-1} \\ T_s &= 2.205 \text{ sec} \\ W_s &= 0.6161 \text{ radians/sec} \\ \xi &= 0.4230 \\ V &= 157.8 \text{ ft./sec} \end{aligned}$$

The constants for equations (4.7) are calculated from (5.1) with reference to figure 2. ILS weather minimums of visibility vary from a half to a quarter mile. Using this as a basis, a value of one-half mile is chosen for D. In consistent units,

$$(5.2) \qquad D = 2640 \text{ ft.}$$

Then from figure 2,

$$(5.3) \qquad h_d(0) = (D + 1000)\tan 3^\circ = 190.8 \text{ ft.}$$

Using the small angle assumption,

$$(5.4) \qquad \dot{h}_d(0) = \gamma_d(0)V$$

where $\gamma_d(0) = -0.05236$ radians

and

$$(5.5) \qquad \begin{aligned} t_1 &= \frac{D}{V} \text{ sec} \\ t_f &= \frac{(D + 1000)}{V} \text{ sec} \end{aligned}$$

A value of twenty feet is chosen for the desired altitude over the end of the runway, then

(5.6) $h_d(t_1) = 20$ ft.

$\gamma_d(t_1) = -0.02$ radians

$\dot{h}_d(t_1) = \gamma_d(t_1)V = -3.156$ ft./sec

The physical limitations of the aircraft must be considered; important details are discussed below.

1) Angle of Attack. The T-28 aircraft stalls at an angle of attack, $\alpha(t)$, of approximately 21° and a speed of 72 knots. As the angle of attack approaches this limit, the speed cannot be maintained constant and the linearized system equations no longer represent the aircraft motion. It is assumed that realistic representation is lost for $\alpha(t) \geq 18^\circ$. This value will be called the stall angle of attack, α_s. For the T-28 in landing configuration descending on an ILS glideslope at 93.5 knots, the angle of attack, $\alpha(0)$, is 11° or 0.1920 radians.

2) Pitch Angle. The pitch angle at touchdown must be between the limits $0^\circ \leq \theta(t_f) \leq 14^\circ$. The lower limit prevents the nose wheel from touching first and the upper limit prevents the tail skag from dragging on the runway.

3) Elevator Control. The elevator motion is limited by the mechanical stops of the actuator arm. Assumed values are -35° and $+15^\circ$. The direction is defined in figure 1. Aircraft motion with the elevator against these stops for non-zero time is not permitted for the linearized model used.

From figure 1,

(5.7) $\theta(t) = \alpha(t) + \gamma(t)$

and the desired pitch angle at t = 0 is

(5.8) $$\theta_d(0) = \alpha_d(0) + \gamma_d(0) = 0.1396 \text{ radians.}$$

Since the angle of attack must remain below the stall value, α_s, an equation for $\alpha(t)$ is desired. From (3.2), the change in pitch rate is

(5.9) $$\Delta\dot{\theta}(t) = \frac{T_s}{V}\Delta\dddot{h}(t) + \frac{1}{V}\Delta\ddot{h}(t).$$

(5.7) may be written in the form

(5.10) $$\theta_e(t) + \Delta\theta(t) = \alpha_e(t) + \Delta\alpha(t) + \gamma_e(t) + \Delta\gamma(t).$$

Taking the time derivative with $\dot{\theta}_e(t) = \dot{\alpha}_e(t) = \dot{\gamma}_e(t) = 0$,

(5.11) $$\Delta\dot{\theta}(t) = \Delta\dot{\alpha}(t) + \Delta\dot{\gamma}(t)$$

and equating (5.9) and (5.11) gives

(5.12) $$\Delta\dot{\alpha}(t) + \Delta\dot{\gamma}(t) = \frac{T_s}{V}\Delta\dddot{h}(t) + \frac{1}{V}\Delta\ddot{h}(t).$$

It can be shown [3] that $\Delta\alpha(t) = T_s\Delta\dot{\gamma}$ so the T_s term of (5.12) is associated with $\Delta\dot{\alpha}(t)$ and

(5.13) $$\Delta\dot{\alpha}(t) = \frac{T_s}{V}\Delta\dddot{h}(t)$$

(5.14) $$\Delta\dot{\gamma}(t) = \frac{1}{V}\Delta\ddot{h}(t).$$

Integrating equations (5.13) and (5.14) yields

(5.15) $$\alpha(t) = \alpha_e(t) + \frac{T_s}{V}\Delta\ddot{h}(t)$$

(5.16) $$\gamma(t) = \gamma_e(t) + \frac{1}{V}\Delta\dot{h}(t)$$

where $\alpha_e(t) = \alpha(0)$ and $\gamma_e(t) = \gamma(0)$.

From (5.4), $\gamma(0) = \dfrac{\dot{h}(0)}{V}$ and (5.16) becomes

(5.17) $$\gamma(t) = \frac{1}{V}x_2(t)$$

Using (5.7), $\alpha(t)$ can now be determined from the system model by

$$(5.18) \qquad \alpha(t) = x_3(t) - \frac{1}{V} x_2(t)$$

In section 2, it was stated that if the measurable signals, $\underline{x}(o)$, were outside specified limits, the aircraft would be waved off for another approach. These limits are somewhat arbitrary and should be chosen only after a close study of existing conditions of actual aircraft making ILS approaches.

For this study the following limits are assumed:

$$(5.19) \qquad \begin{aligned} x_1(0) &= x_{1d}(0) \pm 20 \text{ ft.} \\ x_2(0) &= x_{2d}(0) \pm 2 \text{ ft./sec} \\ x_3(0) &= x_{3d}(0) \pm 0.0350 \text{ radians} \\ x_4(0) &= x_{4d}(0) \pm 0.0043 \text{ radians} \end{aligned}$$

6. MODEL TESTING

Every mathematical model derived for a physical system should be tested by applying known controls to insure that the model accurately describes the system. Originally the $\dot{\underline{z}}(t)$ equations were used as the aircraft model assuming that they represented the signals, $\underline{x}(t)$. In this problem there are two controls that can be used for testing. The control, $u(t) = 0$, applied to the model allows the aircraft to follow its equilibrium flight path which is a constant rate of descent to the runway. From (3.4) a control can be derived using the equations, $\underline{x}_d(t)$, since they are continuous and known functions of time.

$$(6.1) \qquad u(t) = \frac{1}{K_s V W_s^2} \Delta h_d^{IV}(t) + \frac{2\xi}{K_s V W_s} \Delta \dddot{h}_d(t) + \frac{1}{K_s V} \Delta \ddot{h}_d(t)$$

Since $\ddot{h}_e(t)$ and $\dddot{h}_e(t)$ equal zero,

$$(6.2) \qquad \Delta\ddot{h}_d(t) = \ddot{h}_d(t)$$

$$\Delta\dddot{h}_d(t) = \dddot{h}_d(t)$$

Using (4.1) and (4.2),

$$(6.3) \qquad \Delta\ddot{h}_d(t) = 12a_4t^2 + 20a_5t^3 + 30a_6t^4 + 40a_7t^5$$

$$\Delta\dddot{h}_d(t) = 24a_4t + 60a_5t^2 + 120a_6t^3 + 210a_7t^4$$

$$\Delta h_d^{IV}(t) = 24a_4 + 120a_5t + 360a_6t^2 + 840a_7t^3$$

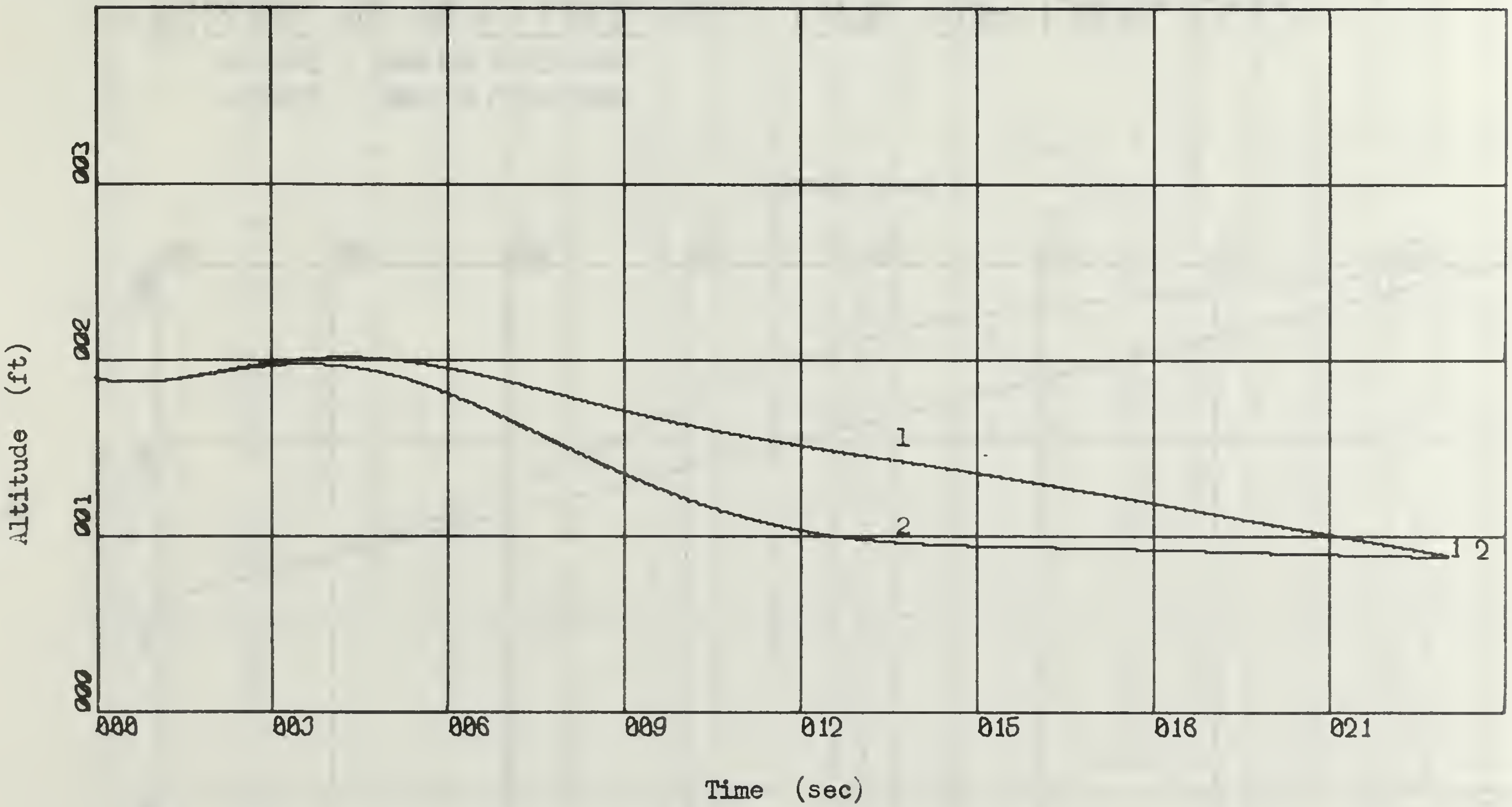
for $0 \leq t \leq t_1$.

$$(6.4) \qquad \Delta\ddot{h}_d(t) = \Delta\dddot{h}_d(t) = \Delta h_d^{IV}(t) = 0$$

for $t_1 \leq t \leq t_f$.

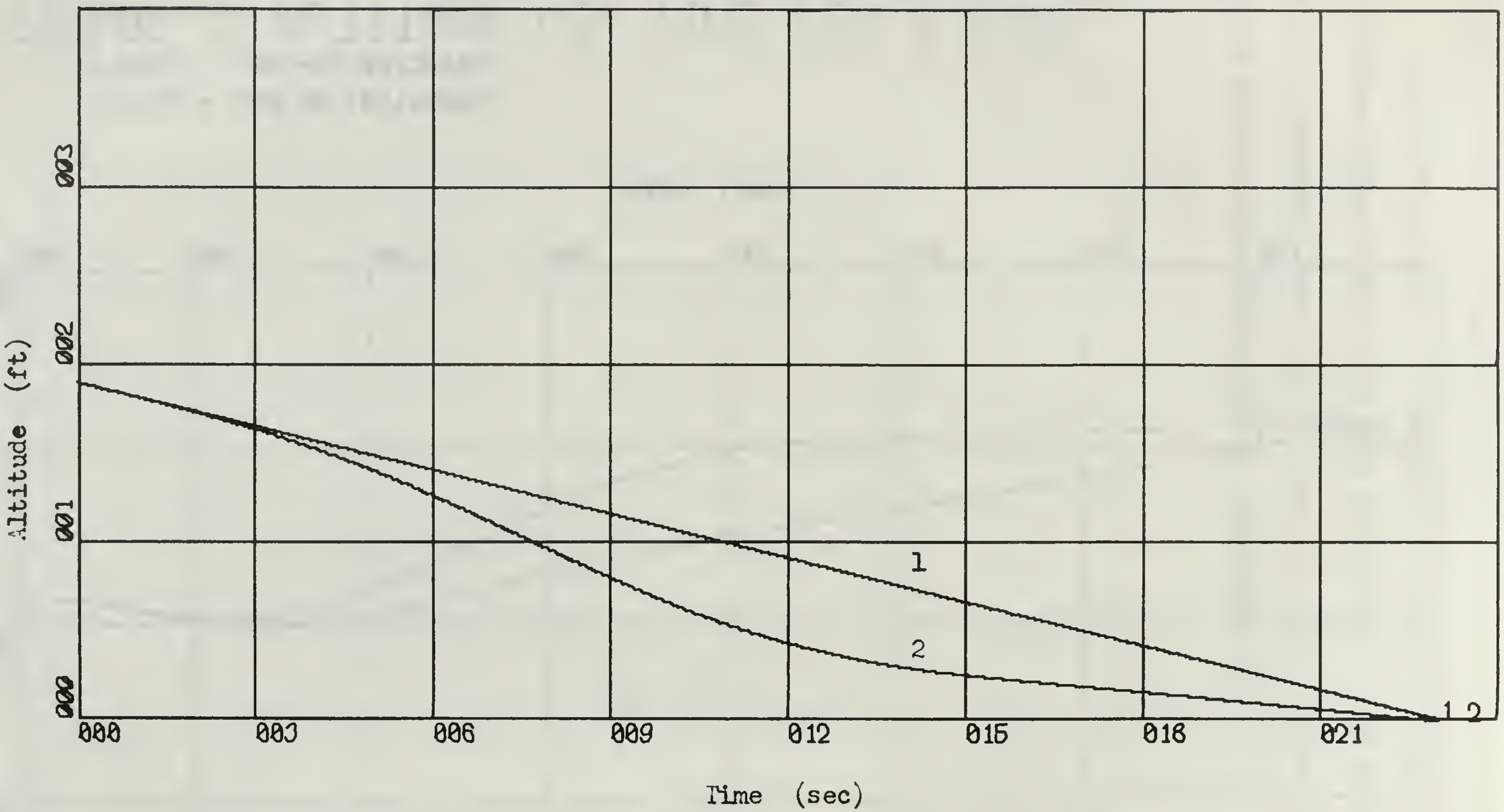
With (6.3) and (6.4) used in (6.1) over their respective time intervals and this u(t) applied to the model, it should follow the desired flight path shown in figure 2.

When these controls were applied to the first model, it did not follow either flight path (see figure 3). After deriving the model in section 3, the same two controls were applied and the proper flight paths were followed (see figure 4).



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 1.00E+02 UNITS/INCH.

FIGURE 3. ALTITUDE VS. TIME FOR $\dot{Z}$ EQNS.
1 IS CONTROL OF ZERO. 2 IS CONTROL OF EQN 6.1



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 1.00E+02 UNITS/INCH.

FIGURE 4. ALTITUDE VS. TIME FOR EQNS. 3.12
1 IS CONTROL OF ZERO. 2 IS CONTROL OF EQN 6.1

7. APPLICATION OF OPTIMIZATION THEORY

The first step in applying Dynamic Programming is to obtain an error index. This index determines the form of the resulting optimal control. The errors, $\underline{x}(a) - \underline{x}_d(a)$, for $0 \leq a \leq t_f$; the errors at the final time, $\underline{x}(t_f) - \underline{x}_d(t_f)$; and the control, u(t), are considered to be significant for this problem. The Bolza form of error index is considered adequate to insure satisfactory system performance. The error index, J, is defined by

$$(7.1) \quad J = \frac{1}{2}\left\{[\underline{x}(t_f) - \underline{x}_d(t_f)]^T F[\underline{x}(t_f) - \underline{x}_d(t_f)]\right\} + \frac{1}{2}\int_o^{t_f}\left\{[\underline{x}(t) - \underline{x}_d(t)]^T Q(t)\,[\underline{x}(t) - \underline{x}_d(t)] + u^2(t)\right\} dt$$

where F is a 4x4, positive, semi-definite, constant matrix.

Q(t) is a 4x4, positive, semi-definite, time variant matrix.

Both F and Q(t) are arbitrary weighting functions of the system errors and are chosen to be symmetric.

The condition by which (7.1) is a minimum is given by [1]

$$(7.2) \quad \min_{u(a)}\left[\frac{1}{2}\left\{[\underline{x}(a) - \underline{x}_d(a)]^T Q(a)\,[\underline{x}(a) - \underline{x}_d(a)] + u^2(a)\right\} + \frac{dE}{da}\right] = 0$$

where a is an arbitrary instant of time in the interval, $0 \leq a \leq t_f$. t_f is the fixed final time. E is defined as the minimum error function, $E[f(a), \underline{x}(a)] = \min\limits_{u(a)} J$, which is assumed to be only a function of a and the signals measured at a.

As $a \longrightarrow t_f$, it can be shown that

$$(7.3) \quad E[f(t_f), \underline{x}(t_f)] = \frac{1}{2}\left\{[\underline{x}(t_f) - \underline{x}_d(t_f)]^T F[\underline{x}(t_f) - \underline{x}_d(t_f)]\right\}$$

Define

$$(7.4) \quad H[\underline{x}(a), u(a)] = \frac{1}{2}\left\{[\underline{x}(a) - \underline{x}_d(a)]^T Q(a)[\underline{x}(a) - \underline{x}_d(a)] + u^2(a)\right\}$$

then (7.2) can be written as

$$(7.5) \quad \min_{u(a)} \left[H[\underline{x}(a), u(a)] + \frac{dE}{da} \right] = 0$$

The derivative of E with respect to a may be written as

$$(7.6) \quad \frac{dE}{da} = \frac{d\, E[f(a), x(a)]}{da}$$

$$= \frac{\partial E}{\partial f(a)} \frac{df(a)}{da} + \sum_{n=1}^{4} \frac{dx_n}{da} \frac{\partial E[f(a), x(a)]}{\partial x_n(a)}$$

The minimal error condition (7.2) can then be written

$$(7.7) \quad \min_{u(a)} \left[H[\underline{x}(a), u(a)] + \frac{\partial E}{\partial f(a)} \frac{df(a)}{da} + \dot{x}_1(a) \frac{\partial E}{\partial x_1} + \right.$$

$$\left. \dot{x}_2(a) \frac{\partial E}{\partial x_2} + \dot{x}_3(a) \frac{\partial E}{\partial x_3} + \dot{x}_4(a) \frac{\partial E}{\partial x_4} \right] = 0$$

For (7.7) to be a minimum for all a, $0 \leq a < t_f$, the first derivative of the term in brackets with respect to u(a) must be zero and the second derivative must be positive. E was defined as a function of a and $\underline{x}(a)$ and is not a function of u(a). From the system equations (3.12) it is seen that $\dot{x}_4(a)$ is the only time derivative that is a function of u(a). The derivatives of (7.7) with respect to u(a) can then be written

$$(7.8) \quad \frac{\partial H}{\partial u(a)} + \frac{\partial E}{\partial x_4} \frac{\partial \dot{x}_4(a)}{\partial u(a)} = 0$$

$$\frac{\partial^2 H}{\partial u(a)^2} + \frac{\partial E}{\partial x_4} \frac{\partial^2 \dot{x}_4(a)}{\partial u(a)^2} > 0$$

From (3.12) and (7.4)

$$(7.9) \quad \frac{\partial \dot{x}_4(a)}{\partial u(a)} = b_4, \quad \frac{\partial^2 \dot{x}_4(a)}{\partial u(a)^2} = 0$$

$$\frac{\partial H}{\partial u(a)} = u(a), \quad \frac{\partial^2 H}{\partial u(a)^2} = 1$$

Substitution of (7.9) in (7.8) yields

$$(7.10)\quad u(a) = -b_4 \frac{\partial E}{\partial x_4}$$

$1 > 0$

(7.10) defines the optimal control for this system over the interval, $0 \leq a < t_f$. Substituting (7.4), (3.12), and (7.10) in (7.7) gives

$$(7.11)\quad \frac{1}{2}\left\{[\underline{x}(a) - \underline{x}_d(a)]^T Q(a)[\underline{x}(a) - \underline{x}_d(a)]\right\} + \frac{b_4^2}{2}\left[\frac{\partial E}{\partial x_4}\right]^2 +$$

$$\frac{\partial E}{\partial f(a)}\frac{df(a)}{da} + x_2(a)\frac{\partial E}{\partial x_1} + \Big[a_{22}[x_2(a) - x_2(0)] +$$

$$a_{23}[x_3(a) - x_3(0)]\Big]\frac{\partial E}{\partial x_2} + x_4(a)\frac{\partial E}{\partial x_3} + \Big[a_{42}[x_2(a) - x_2(0)]$$

$$+ a_{43}[x_3(a) - x_3(0)] + a_{44}x_4(a)\Big]\frac{\partial E}{\partial x_4} - b_4^2\left[\frac{\partial E}{\partial x_4}\right]^2 = 0$$

which is a partial differential equation defining E. If (7.11) can be solved for E, then the optimal control of (7.10) can be found.

The Parametric Expansion Method [2] is now applied to solve (7.11) and provide an optimal solution. The method is outlined as follows:

Assume a form for E consistent with the functional description.

$$(7.12)\quad E = k(a) - 2\sum_{m=1}^{4} k_m(a)\, x_m(a) + \sum_{m=1}^{4}\sum_{p=1}^{4} k_{mp}(a)x_m(a)x_p(a)$$

where $k_{mp}(a) = k_{pm}(a)$.

Taking the derivative with respect to $x_4(a)$, substitution in (7.10) yields the optimal control as a function of time, the aircraft parameters, and the measurable signals, $\underline{x}(a)$.

$$(7.13)\quad u(t) = 2b_4[k_4(t) - k_{14}(t)x_1(t) - k_{24}(t)x_2(t)$$

$$- k_{34}(t)x_3(t) - k_{44}(t)x_4(t)]$$

since a is an arbitrary instant, it can be real time, t. Obtain the derivatives required by (7.11) from (7.12). Substitute these in (7.11) and collect terms so that the expression is in the form:

$$(7.14)\quad P_o + P_1x_1(a) + P_2x_2(a) + P_3x_3(a) + P_4x_4(a) + P_{11}x_1^2(a) + P_{22}x_2^2(a) + P_{33}x_3^2(a) + P_{44}x_4^2(a) + P_{12}x_1(a)x_2(a) + P_{13}x_1(a)x_3(a) + P_{14}x_1(a)x_4(a) + P_{23}x_2(a)x_3(a) + P_{24}x_2(a)x_4(a) + P_{34}x_3(a)x_4(a) = 0$$

The P coefficients contain terms of the k-parameters, time derivatives of the k(a)'s, the desired signals, the weighting factors, the aircraft constants and the chosen equilibrium conditions. For (7.14) to be valid for all values of the measured signals, each P coefficient must independently equal zero.

$$(7.15)\quad P_o = P_m = P_{mp} = 0$$

where $m = 1, 2, 3, 4$; $p = 1, 2, 3, 4$.

Equation (7.15) gives a set of 15 independent, ordinary, first-order differential equations which define the k(a)'s. These equations with the boundary conditions are derived in Appendix B.

8. IMPLEMENTATION OF A CONTROLLER

Several types of controllers could be designed depending on the assumed form of E and the methods available for measuring time and velocity. The block diagram shown in figure 5 is obtained from (7.13) and is a simple analog control system. A constant speed motor rotates the shaft at a constant velocity. The k(t)'s of (7.13) represent the settings of wire-wound potentiometers where the wiper arms rotate with the shaft. The measured signals are multiplied by the appropriate k(t), summed and multiplied to provide the elevator deflection signal.

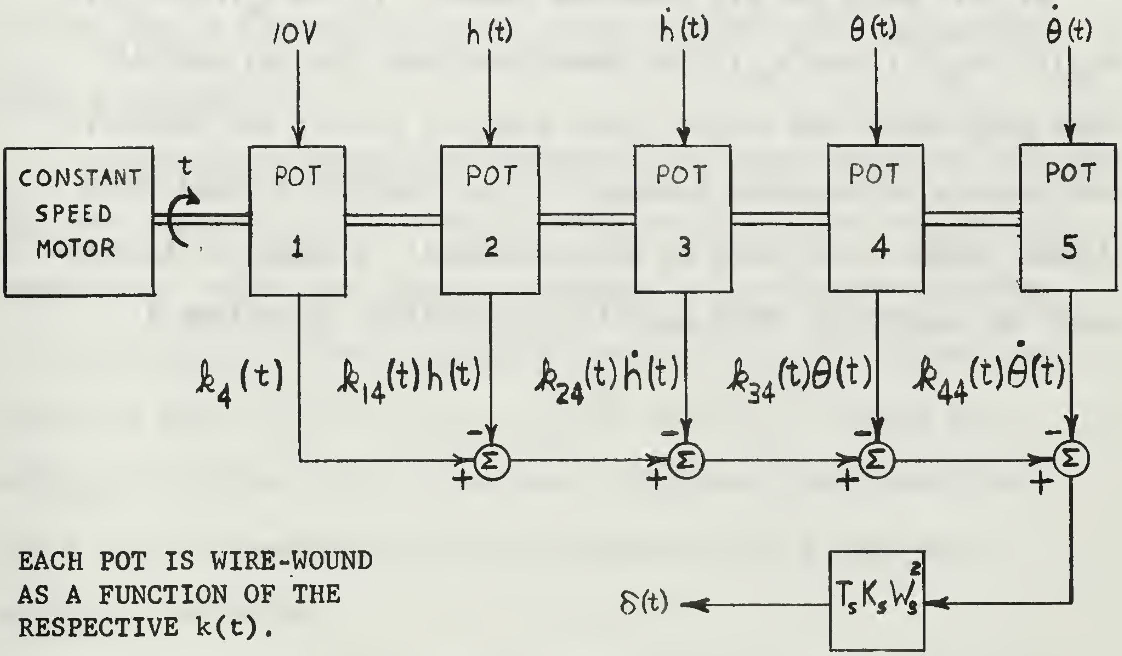


Figure 5. Elevator Controller

In the derivation of the k(t) equations, the equilibrium values of the descent rate, $x_{2e}(t)$, and pitch angle, $x_{3e}(t)$, must be known. This means that the optimal k(t) equations are dependent on the equilibrium conditions. This is a function of the aircraft model and would be altered with a different aircraft model. Little error should be introduced if these values are assumed to be the desired descent rate and pitch angle at problem start.

$$(8.1) \qquad x_{2e}(t) = x_{2d}(0)$$

$$x_{3e}(t) = x_{3d}(0)$$

By assuming these values the system is no longer optimal, but it is desired to sacrifice optimality in the interest of simplicity and to investigate the feasibility of this simple controller. The k(t) equations are also dependent on the aircraft constants so a different set must be obtained for each type of aircraft.

At this point two big questions remain. 1) Can $k_4(t)$, $k_{14}(t)$, $k_{24}(t)$, $k_{34}(t)$, and $k_{44}(t)$ be found such that, for all aircraft with $\underline{x}(0)$ within the signal limits given by (5.19), the aircraft will make a satisfactory landing? 2) Are the k(t)'s found above linear enough to be wound on potentiometers? A method to investigate the answers to these questions is outlined in section 9.

9. INVESTIGATION PROCEDURE

Since all of the constants and variables of the $\dot{k}(t)$ equations, (B.5), have been calculated or are known except the weighting factors F and Q(t), the answers to the questions in section 8 can be investigated by the following procedure:

1) Arbitrarily pick the weighting factors of the error index, F and Q(t).

2) Integrate equations (B.5) from $t = t_f$ to $t = 0$ to obtain the k(t)'s of (7.13).

3) Integrate the aircraft model equations (3.12) from $t = 0$ to $t = t_f$ using u(t) of (7.13) with the k(t)'s obtained in 2.

4) Observe the final signal errors, $\underline{x}(t_f) - \underline{x}_d(t_f)$. If these are not within prescribed limits, change F and/or Q(t) and repeat steps 2 through 4.

A program was written to accomplish the above procedure using the CDC 1604 computer and an existing modified Runge-Kutta integration subroutine. There were two main sections in the computer program. 1) Equations (B.5) were integrated from $t = t_f$ to $t = 0$ with the values of $k_4(t)$, $k_{14}(t)$, $k_{24}(t)$, $k_{34}(t)$, and $k_{44}(t)$ stored in a buffer. 2) Three sets of equations (3.12) were integrated from $t = 0$ to $t = t_f$ using u(t) of (7.13) and the k(t)'s that were stored in the buffer.

Three aircraft models were chosen for each program so that one could be started with positive errors, one started on the desired trajectory, and the third started with negative errors. This allowed all of the system errors to be observed in one program.

The final error limits were arbitrarily chosen as:

a) An altitude, descent rate combination such that all three aircraft land within ± 500 feet of the desired touchdown point with a descent rate of $\pm$ 100 ft./min or $\pm$ 1.66 ft./sec from the desired rate of descent, $x_{2d}(t_f)$.

b) Pitch angle as stated in section 5, $0^\circ \leq \theta(t_f) \leq 14^\circ$.

c) No limit was set on the pitch rate but it must be small for the approach to closely follow the desired flight path.

For this investigation, the aircraft was considered to have only an altitude error and $Q(t) = Q$, a constant matrix consisting of the diagonal elements only. The study was conducted in the following manner:

1) Determine if successful approaches can be made with constant weighting of any one signal; or any two signals; or combinations of any three signals; and finally, combinations of all four signals.

2) Determine if successful approaches can be made with constant weighting of all signals, progressively adding diagonal elements of the F matrix.

3) When successful approaches are found, determine the nature of the k(t)'s required for the controller.

10. RESULTS

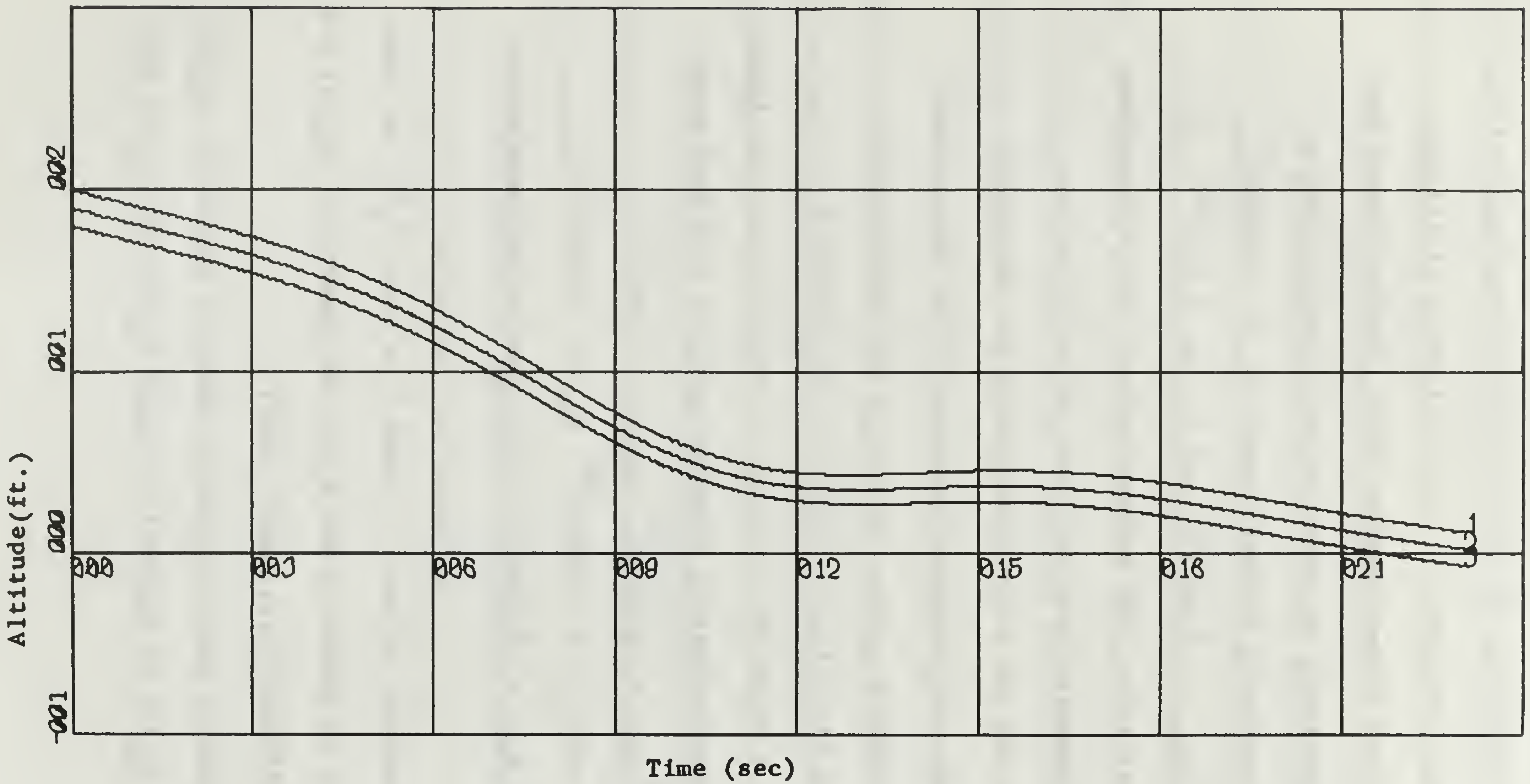
Digital simulation was used to study the areas outlined in section 9. The results were:

1) Constant weighting of all four signals was required to obtain a satisfactory trajectory. All final error limits were met except that of the minimum descent rate. Figure 6 shows the altitude vs. time/distance for three aircraft started with an initial altitude error of $\pm$ 10 feet, where $q_{11} = 2.5\text{x}10^{-10}$, $q_{22} = 8.5\text{x}10^{-10}$, $q_{33} = 6.0\text{x}10^{-5}$, and $q_{44} = 8.8\text{x}10^{-4}$. All other q's and F were zero. The minimum descent rate at touchdown was -5.36 ft./sec compared with the limit of -4.91 ft./sec.

2) This area was not fully investigated but successful approaches were made with constant weighting of the four signals (Q equal to zero except q_{11}, q_{22}, q_{33}, q_{44}) and weighting all four signals at the final time (F equal to zero except f_{11}, f_{22}, f_{33}, f_{44}). Figure 7 shows the altitude vs. time/distance for three aircraft started with an initial altitude error of $\pm$ 20 feet where $q_{11} = 7.10\text{x}10^{-10}$, $q_{22} = 2.95\text{x}10^{-7}$, $q_{33} = 7.00\text{x}10^{-5}$, $q_{44} = 8.00\text{x}10^{-4}$ and $f_{11} = 1.80\text{x}10^{-9}$, $f_{22} = 2.00\text{x}10^{-7}$, $f_{33} = 2.90\text{x}10^{-3}$, $f_{44} = 3.00\text{x}10^{-3}$. All final errors were within specified limits.

3) For the approach of part 1, where F is zero, $k_4(t)$ is shown in figure 8, $k_{34}(t)$ in figure 9, and $k_{44}(t)$ in figure 10. $k_{14}(t)$ and $k_{24}(t)$ were insignificant (i.e. nearly zero).

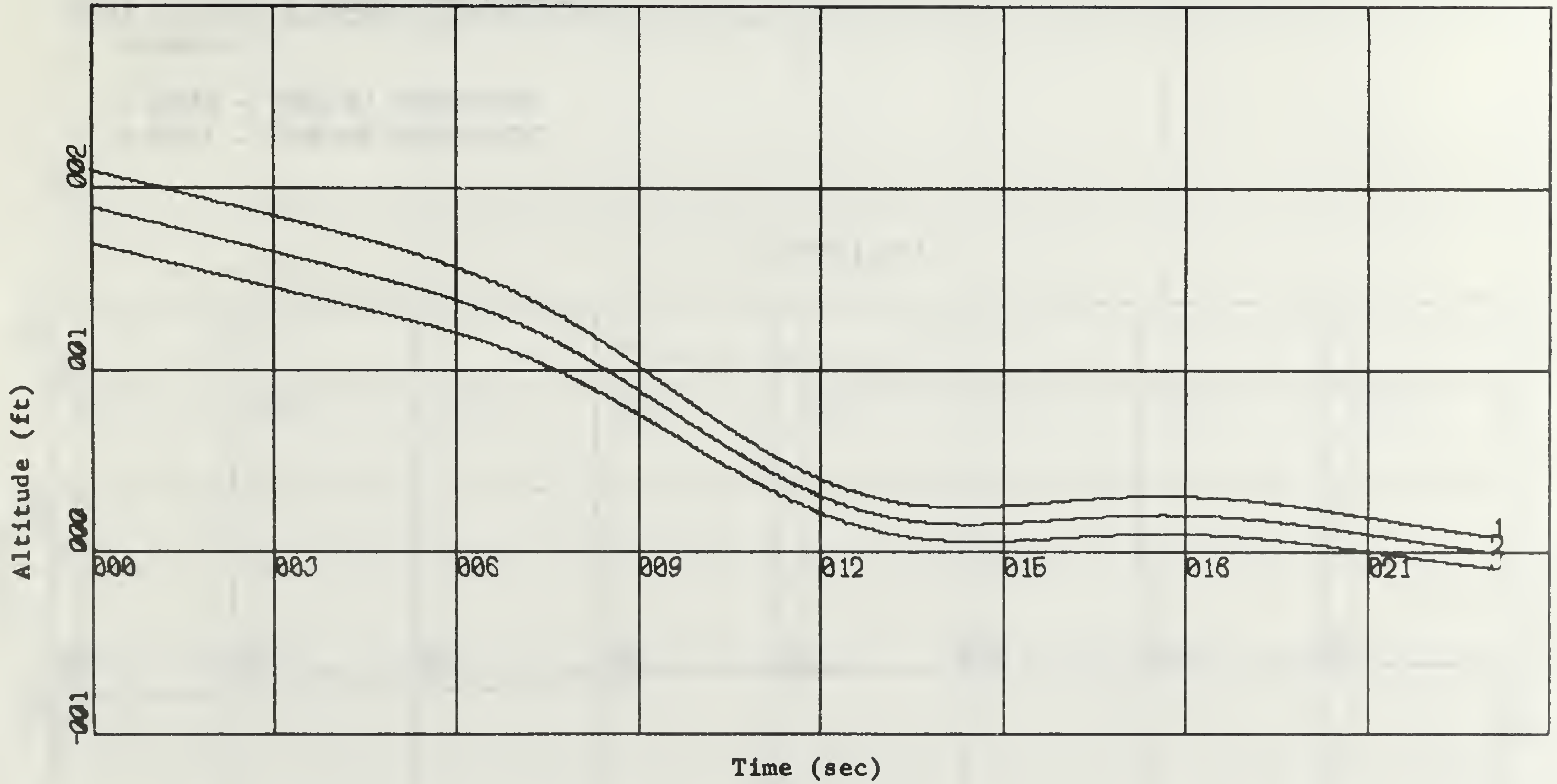
For the approach of part 2, $k_4(t)$ is shown in figure 11, $k_{34}(t)$ in figure 12, and $k_{44}(t)$ in figure 13. Again $k_{14}(t)$ and $k_{24}(t)$ were insignificant.



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 1.00E+02 UNITS/INCH.

FIGURE 6.

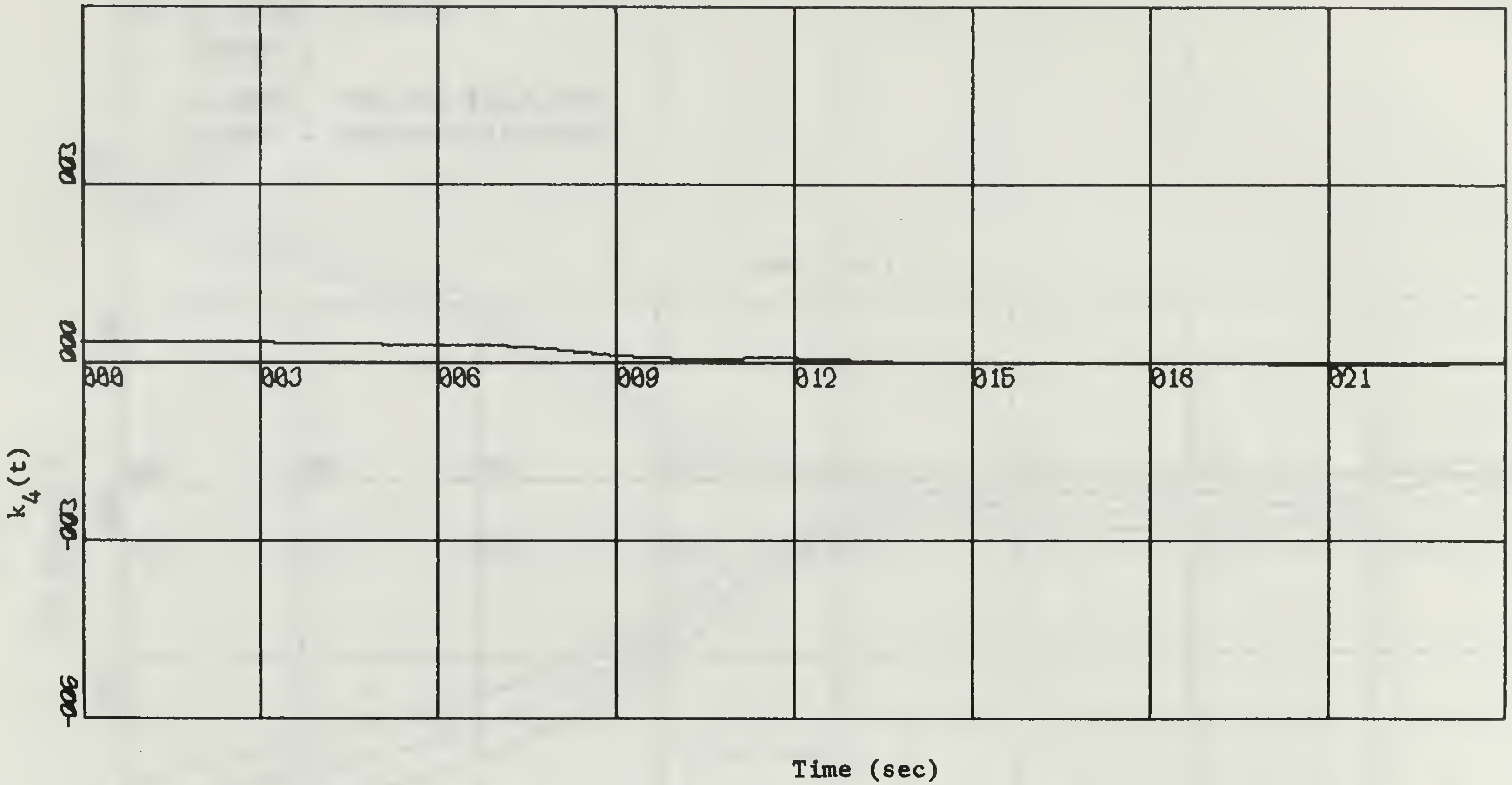
ALT VS. TIME FOR Q ONLY



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 1.00E+02 UNITS/INCH.

FIGURE 7.

ALT VS. TIME FOR F AND Q



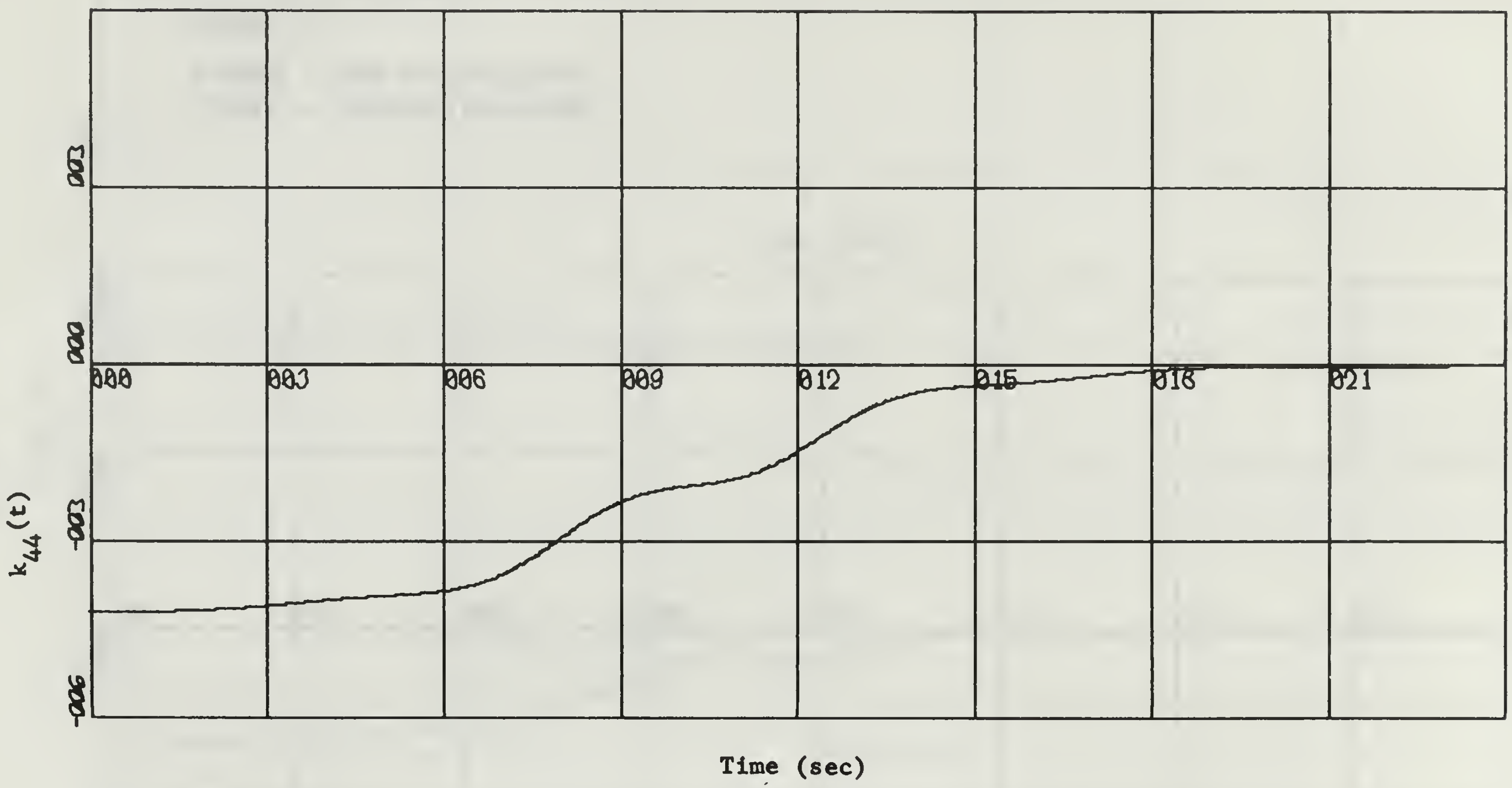
X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 3.00E-01 UNITS/INCH.

FIGURE 8.
K4 VS. TIME FOR Q ONLY

X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 3.00E-01 UNITS/INCH.

FIGURE 9.

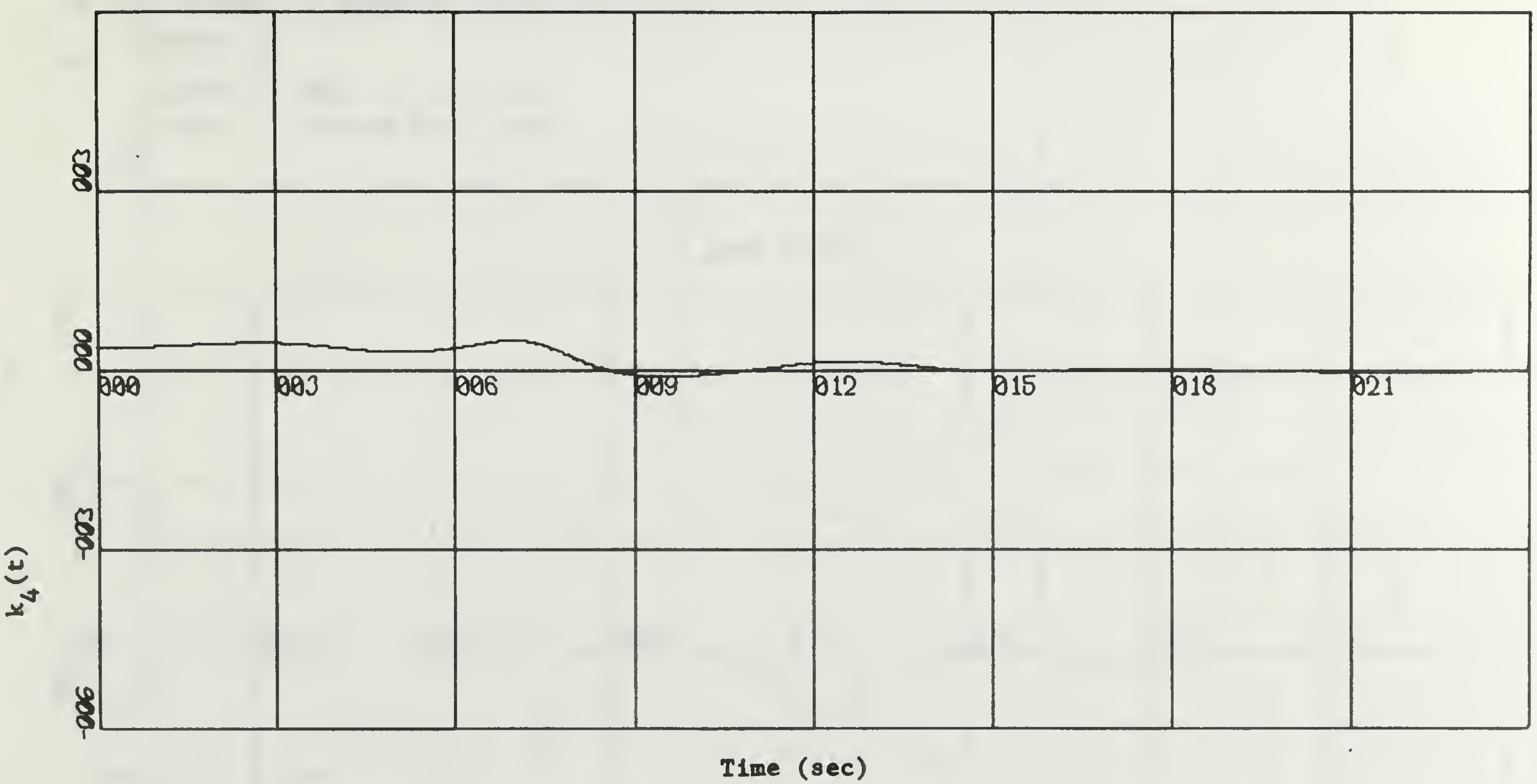
K34 VS. TIME FOR Q ONLY



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 3.00E-01 UNITS/INCH.

FIGURE 10.

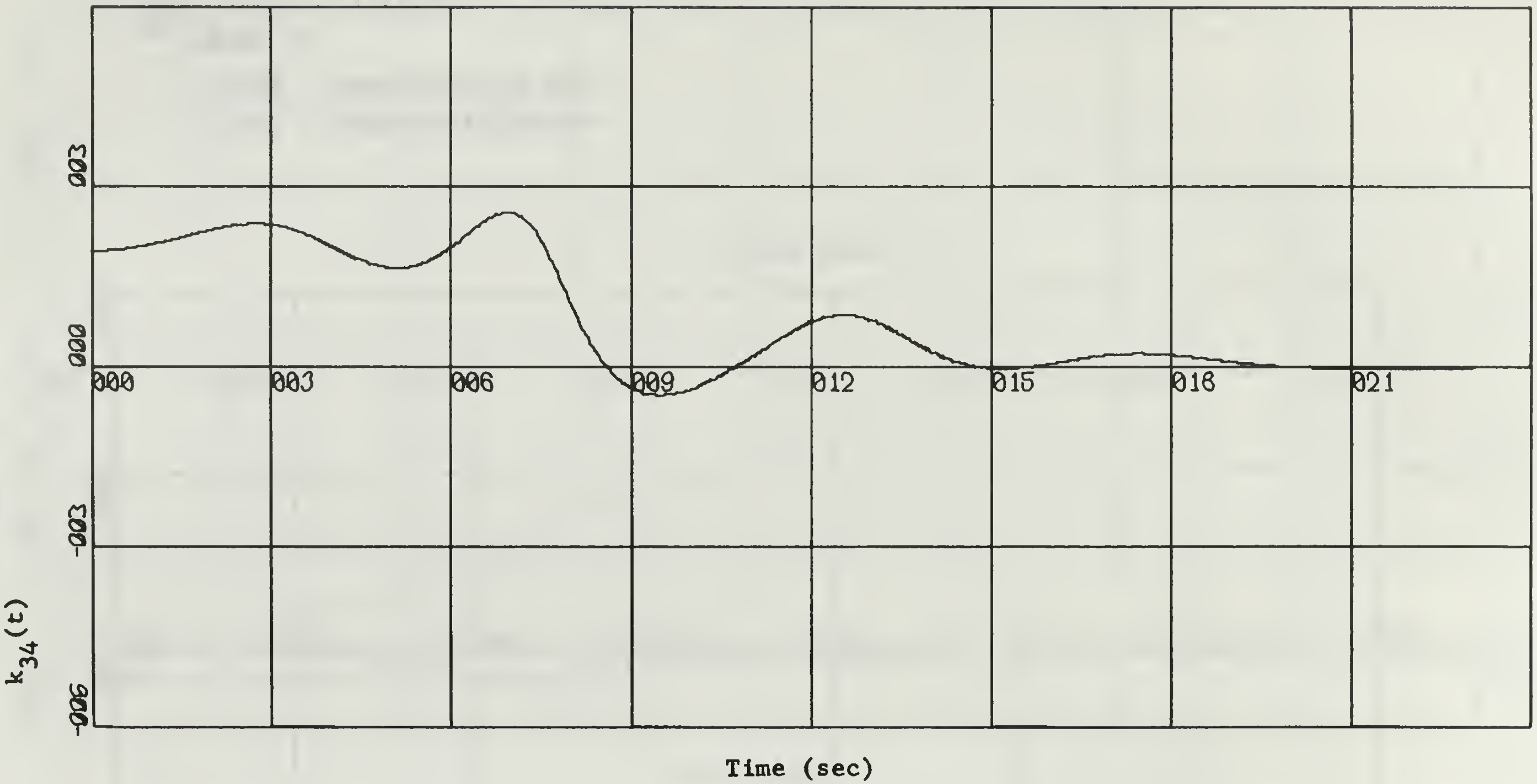
K44 VS. TIME FOR Q ONLY.



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 3.00E-01 UNITS/INCH.

FIGURE 11.

K4 VS. TIME FOR F AND Q



X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 3.00E-01 UNITS/INCH.

FIGURE 12.

K34 VS. TIME FOR F AND Q

X-SCALE = 3.00E+00 UNITS/INCH.
Y-SCALE = 3.00E-01 UNITS/INCH.
FIGURE 13.
K44 VS. TIME FOR F AND Q

11. CONCLUSIONS

For the results specified in section 10, the following conclusions are stated:

1) Although the minimum descent rate error was not within limits using the Q matrix only, it is felt that with several small adjustments of the weighting factors the descent rate could be brought within limits. The k(t)'s are of the same sign with maximum values less than one and could be wound on the potentiometers.

2) While all of the error limits were met for the approach using both F and Q, the k(t)'s changed sign and are unsatisfactory for this controller. As adjustments were made to F and Q, bringing the flight paths closer to the desired trajectory, the k(t)'s approached the same sign over the interval of flight. It is believed that if F and Q are adjusted for flight paths with smaller errors, this problem would be eliminated.

Two recommendations are made:

1) That investigation be continued until definite conclusions can be obtained about the practicality of this controller.

2) That a method be investigated to systemmatically adjust the weighting factors, the elements of the Q and F matrices.

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APPENDIX A. DERIVATION OF $\dot{\underline{z}}(t)$ EQUATIONS

Define

$$\text{(A.1)}\quad \begin{aligned} z_1(t) &= \Delta h(t) \\ z_2(t) &= \Delta \dot{h}(t) \\ z_3(t) &= \Delta\theta(t) \\ z_4(t) &= \Delta\dot{\theta}(t) \\ u(t) &= \Delta\delta(t) \end{aligned}$$

The time derivatives of the z(t)'s are:

$$\text{(A.2)}\quad \begin{aligned} \dot{z}_1(t) &= \Delta\dot{h}(t) = z_2(t) \\ \dot{z}_2(t) &= \Delta\ddot{h}(t) \\ \dot{z}_3(t) &= \Delta\dot{\theta}(t) = z_4(t) \\ \dot{z}_4(t) &= \Delta\ddot{\theta}(t) \end{aligned}$$

Equations for $\Delta\ddot{h}(t)$ and $\Delta\ddot{\theta}(t)$ in terms of the states, $\underline{z}(t)$, are needed. (3.2) can be written as

$$\text{(A.3)}\quad T_s\Delta\dddot{h}(s) + \Delta\ddot{h}(s) = V\Delta\dot{\theta}(s)$$

Taking the inverse Laplace transform and solving for $\Delta\dddot{h}(t)$ yields

$$\text{(A.4)}\quad \Delta\dddot{h}(t) = -\frac{1}{T_s}\Delta\ddot{h}(t) + \frac{V}{T_s}\Delta\dot{\theta}(t)$$

Differentiating (A.4) gives

$$\text{(A.5)}\quad \Delta h^{IV}(t) = -\frac{1}{T_s}\Delta\dddot{h}(t) + \frac{V}{T_s}\Delta\dot{\theta}(t)$$

Integrating (A.4) and assuming $\Delta\dddot{h}(0) = 0$ gives

$$\text{(A.6)}\quad \Delta\ddot{h}(t) = -\frac{1}{T_s}\Delta\dot{h}(t) + \frac{V}{T_s}\Delta\theta(t)$$

Substituting equations (A.4), (A.5) and (A.6) in (3.2) and solving for $\Delta\ddot{\theta}(t)$ gives

(A.7) $$\Delta\dot{\theta}(t) = \frac{1}{V}\left[\frac{1}{T_s^2} - \frac{2\delta W_s}{T_s} + W_s^2\right]\Delta\dot{h}(t)$$

$$- \left[\frac{1}{T_s^2} - \frac{2\delta W_s}{T_s} + W_s^2\right]\Delta\theta(t) + \left[\frac{1}{T_s} - 2\delta W_s\right]\Delta\dot{\theta}(t)$$

$$+ K_s T_s W_s^2 \Delta\delta(t)$$

Inserting (A.6) and (A.7), written in terms of (A.1), into (A.2) yields

(A.8) $$\dot{z}_1(t) = z_2(t)$$

$$\dot{z}_2(t) = -\frac{1}{T_s} z_2(t) + \frac{V}{T_s} z_3(t)$$

$$\dot{z}_3(t) = z_4(t)$$

$$\dot{z}_4(t) = \frac{1}{V}\left[\frac{1}{T_s^2} - \frac{2\delta W_s}{T_s} + W_s^2\right] z_2(t) - \left[\frac{1}{T_s^2} - \frac{2\delta W_s}{T_s} + W_s^2\right] z_3(t)$$

$$+ \left[\frac{1}{T_s} - 2\delta W_s\right] z_4(t) + K_s T_s W_s^2 u(t)$$

which is in the form

(A.9) $$\dot{\underline{z}}(t) = A\underline{z}(t) + Bu(t)$$

where

$$A \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad B \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}$$

and $a_{22} = -\frac{1}{T_s}$, $a_{23} = \frac{V}{T_s}$, $a_{42} = \frac{1}{V}\left[\frac{1}{T_s^2} - \frac{2\delta W_s}{T_s} + W_s^2\right]$

$$a_{43} = -\left[\frac{1}{T_s^2} - \frac{2\delta W_s}{T_s} + W_s^2\right], \quad a_{44} = \left[\frac{1}{T_s} - 2\delta W_s\right], \quad b_4 = K_s T_s W_s^2$$

APPENDIX B. DERIVATION OF $\dot{k}_{mp}(t)$ EQUATIONS

The partial derivatives of E defined in (7.12) are needed for substitution in (7.11). Since a is an arbitrary instant of time in the interval $0 \leq a \leq t_f$, it can be chosen as real time, t. The functional notation, (t), on all the variables will be dropped for simplicity.

The required partial derivatives of E are:

$$\text{(B.1)} \quad \frac{\partial E}{\partial f(t)} \frac{df(t)}{dt} = \frac{\partial E}{\partial k}\dot{k} + \frac{\partial E}{\partial k_1}\dot{k}_1 + \frac{\partial E}{\partial k_2}\dot{k}_2 + \frac{\partial E}{\partial k_3}\dot{k}_3 + \frac{\partial E}{\partial k_4}\dot{k}_4 + \frac{\partial E}{\partial k_{11}}\dot{k}_{11} + \frac{\partial E}{\partial k_{22}}\dot{k}_{22} + \frac{\partial E}{\partial k_{33}}\dot{k}_{33}$$

$$+ \frac{\partial E}{\partial k_{44}}\dot{k}_{44} + \frac{\partial E}{\partial k_{12}}\dot{k}_{12} + \frac{\partial E}{\partial k_{13}}\dot{k}_{13} + \frac{\partial E}{\partial k_{14}}\dot{k}_{14} + \frac{\partial E}{\partial k_{23}}\dot{k}_{23} + \frac{\partial E}{\partial k_{24}}\dot{k}_{24} + \frac{\partial E}{\partial k_{34}}\dot{k}_{34},$$

$$\frac{\partial E}{\partial k} = 1, \quad \frac{\partial E}{\partial k_1} = -2x_1, \quad \frac{\partial E}{\partial k_2} = -2x_2, \quad \frac{\partial E}{\partial k_3} = -2x_3,$$

$$\frac{\partial E}{\partial k_4} = -2x_4, \quad \frac{\partial E}{\partial k_{11}} = x_1^2, \quad \frac{\partial E}{\partial k_{22}} = x_2^2, \quad \frac{\partial E}{\partial k_{33}} = x_3^2,$$

$$\frac{\partial E}{\partial k_{44}} = x_4^2, \quad \frac{\partial E}{\partial k_{12}} = 2x_1x_2, \quad \frac{\partial E}{\partial k_{13}} = 2x_1x_3, \quad \frac{\partial E}{\partial k_{14}} = 2x_1x_4,$$

$$\frac{\partial E}{\partial k_{23}} = 2x_2x_3, \quad \frac{\partial E}{\partial k_{24}} = 2x_2x_4, \quad \frac{\partial E}{\partial k_{34}} = 2x_3x_4,$$

$$\frac{\partial E}{\partial x_1} = 2(-k_1 + k_{11}x_1 + k_{12}x_2 + k_{13}x_3 + k_{14}x_4),$$

$$\frac{\partial E}{\partial x_2} = 2(-k_2 + k_{12}x_1 + k_{22}x_2 + k_{23}x_3 + k_{24}x_4),$$

$$\frac{\partial E}{\partial x_3} = 2(-k_3 + k_{13}x_1 + k_{23}x_2 + k_{33}x_3 + k_{34}x_4),$$

$$\frac{\partial E}{\partial x_4} = 2(-k_4 + k_{14}x_1 + k_{24}x_2 + k_{34}x_3 + k_{44}x_4),$$

$$\left[\frac{\partial E}{\partial x_4}\right]^2 = 4(k_4^2 - 2k_4k_{14}x_1 - 2k_4k_{24}x_2 - 2k_4k_{34}x_3$$

$$- 2k_4k_{44}x_4 + k_{14}^2x_1^2 + k_{24}^2x_2^2 + k_{34}^2x_3^2 + k_{44}^2x_4^2$$

$$+ 2k_{14}k_{24}x_1x_2 + 2k_{14}k_{34}x_1x_3 + 2k_{14}k_{44}x_1x_4$$

$$+ 2k_{24}k_{34}x_2x_3 + 2k_{24}k_{44}x_2x_4 + 2k_{34}k_{44}x_3x_4)$$

The derivatives of (B.1) are then substituted in (7.11) to obtain

(B.2) $$\frac{1}{2}\left\{(\underline{x} - \underline{x}_d)^TQ(\underline{x} - \underline{x}_d)\right\} - 2b_4^2(k_4^2 - 2k_4k_{14}x_1 - 2k_4k_{24}x_2$$

$$-2k_4k_{34}x_3 - 2k_kk_{44}x_4 + k_{14}^2x_1^2 + k_{24}^2x_2^2 + k_{34}^2x_3^2 +$$

$$k_{44}^2x_4^2 + 2k_{14}k_{24}x_1x_2 + 2k_{14}k_{34}x_1x_3 + 2k_{14}k_{44}x_1x_4 +$$

$$2k_{24}k_{34}x_2x_3 + 2k_{24}k_{44}x_2x_4 + 2k_{34}k_{44}x_3x_4) + \dot{k} - 2\dot{k}_1x_1 -$$

$$2\dot{k}_2x_2 - 2\dot{k}_3x_3 - 2\dot{k}_4x_4 + \dot{k}_{11}x_1^2 + \dot{k}_{22}x_2^2 + \dot{k}_{33}x_3^2 +$$

$$\dot{k}_{44}x_4^2 + 2\dot{k}_{12}x_1x_2 + 2\dot{k}_{13}x_1x_3 + 2\dot{k}_{14}x_1x_4 + 2\dot{k}_{23}x_2x_3 +$$

$$2\dot{k}_{24}x_2x_4 + 2\dot{k}_{34}x_3x_4 + 2x_2(-k_1 + k_{11}x_1 + k_{12}x_2 + k_{13}x_3 +$$

$$k_{14}x_4) + 2[a_{22}(x_2 - x_2(0)) + a_{23}(x_3 - x_3(0))]\,(-k_2 +$$

$$k_{12}x_1 + k_{22}x_2 + k_{23}x_3 + k_{24}x_4) + 2x_4(-k_3 + k_{13}x_1 +$$

$$k_{23}x_2 + k_{33}x_3 + k_{34}x_4) + 2[a_{42}(x_2 - x_2(0)) +$$

$$a_{43}(x_3 - x_3(0)) + a_{44}x_4]\,(-k_4 + k_{14}x_1 + k_{24}x_2 +$$

$$k_{34}x_3 + k_{44}x_4) = 0$$

The matrices in the first term are expanded and all terms are grouped in the form of (7.14). The resulting P coefficients are:

(B.4)
$$P_o = \dot{k} - 2b_4^2 k_4^2 + \frac{1}{2}(q_{11}x_{1d}^2 + q_{22}x_{2d}^2 + q_{33}x_{3d}^2 + q_{44}x_{4d}^2) + q_{12}x_{1d}x_{2d} + q_{13}x_{1d}x_{3d} + q_{14}x_{1d}x_{4d} + q_{23}x_{2d}x_{3d} + q_{24}x_{2d}x_{4d} + q_{34}x_{3d}x_{4d} + 2k_2\left(a_{22}x_2(0) + a_{23}x_3(0)\right) + 2k_4\left(a_{42}x_2(0) + a_{43}x_3(0)\right)$$

$$P_1 = -2\dot{k}_1 + 4b_4^2 k_4 k_{14} - (q_{11}x_{1d} + q_{12}x_{2d} + q_{13}x_{3d} + q_{14}x_{4d}) - 2k_{12}\left(a_{22}x_2(0) + a_{23}x_3(0)\right) - 2k_{14}\left(a_{42}x_2(0) + a_{43}x_3(0)\right)$$

$$P_2 = -2\dot{k}_2 + 4b_4^2 k_4 k_{24} - (q_{12}x_{1d} + q_{22}x_{2d} + q_{23}x_{3d} + q_{24}x_{4d}) - 2k_{22}\left(a_{22}x_2(0) + a_{23}x_3(0)\right) - 2k_{24}\left(a_{42}x_2(0) + a_{43}x_3(0)\right) - 2k_1 - 2a_{22}k_2 - 2a_{42}k_4$$

$$P_3 = -2\dot{k}_3 + 4b_4^2 k_4 k_{34} - (q_{13}x_{1d} + q_{23}x_{2d} + q_{33}x_{3d} + q_{34}x_{4d}) - 2k_{23}\left(a_{22}x_2(0) + a_{23}x_3(0)\right) - 2k_{34}\left(a_{42}x_2(0) + a_{43}x_3(0)\right) - 2a_{23}k_2 - 2a_{43}k_4$$

$$P_4 = -2\dot{k}_4 + 4b_4^2 k_4 k_{44} - (q_{14}x_{1d} + q_{24}x_{2d} + q_{34}x_{3d} + q_{44}x_{4d}) - 2k_{24}\left(a_{22}x_2(0) + a_{23}x_3(0)\right) - 2k_{44}\left(a_{42}x_2(0) + a_{43}x_3(0)\right) - 2k_3 - 2a_{44}k_4$$

$$P_{11} = \dot{k}_{11} + \frac{1}{2}q_{11} - 2b_4^2 k_{14}^2$$

$$P_{22} = \dot{k}_{22} + \frac{1}{2}q_{22} - 2b_4^2 k_{24}^2 + 2k_{12} + 2a_{22}k_{22} + 2a_{42}k_{24}$$

$$P_{33} = \dot{k}_{33} + \frac{1}{2}q_{33} - 2b_4^2 k_{34}^2 + 2a_{23}k_{23} + 2a_{43}k_{34}$$

$$P_{44} = \dot{k}_{44} + \frac{1}{2}q_{44} - 2b_4^2 k_{44}^2 + 2k_{34} + 2a_{44}k_{44}$$

$$P_{12} = 2\dot{k}_{12} + q_{12} - 4b_4^2k_{14}k_{24} + 2k_{11} + 2a_{22}k_{12} + 2a_{42}k_{14}$$

$$P_{13} = 2\dot{k}_{13} + q_{13} - 4b_4^2k_{14}k_{34} + 2a_{23}k_{12} + 2a_{43}k_{14}$$

$$P_{14} = 2\dot{k}_{14} + q_{14} - 4b_4^2k_{14}k_{44} + 2k_{13} + 2a_{44}k_{14}$$

$$P_{23} = 2\dot{k}_{23} + q_{23} - 4b_4^2k_{24}k_{34} + 2k_{13} + 2a_{22}k_{23} + 2a_{23}k_{22} + 2a_{42}k_{34} + 2a_{43}k_{24}$$

$$P_{24} = 2\dot{k}_{24} + q_{24} - 4b_4^2k_{24}k_{44} + 2k_{14} + 2k_{23} + 2a_{22}k_{24} + 2a_{42}k_{44} + 2a_{44}k_{24}$$

$$P_{34} = 2\dot{k}_{34} + q_{34} - 4b_4^2k_{34}k_{44} + 2k_{33} + 2a_{23}k_{24} + 2a_{43}k_{44} + 2a_{44}k_{34}$$

For each P coefficient to equal zero for all values of the measured signals as required by (7.15), the following first order, linear differential equations must be satisfied:

(B.5)

$$\begin{aligned}\dot{k} = {} & 2b_4^2k_4^2 - \frac{1}{2}(q_{11}x_{1d}^2 + q_{22}x_{2d}^2 + q_{33}x_{3d}^2 + q_{44}x_{4d}^2) \\ & - q_{12}x_{1d}x_{2d} - q_{13}x_{1d}x_{3d} - q_{14}x_{1d}x_{4d} - q_{23}x_{2d}x_{3d} \\ & - q_{24}x_{2d}x_{4d} - q_{34}x_{3d}x_{4d} - 2k_2\left(a_{22}x_2(0) + a_{23}x_3(0)\right) \\ & - 2k_4\left(a_{42}x_2(0) + a_{43}x_3(0)\right)\end{aligned}$$

$$\begin{aligned}\dot{k}_1 = {} & 2b_4^2k_4k_{14} - \frac{1}{2}(q_{11}x_{1d} + q_{12}x_{2d} + q_{13}x_{3d} + q_{14}x_{4d}) \\ & - k_{12}\left(a_{22}x_2(0) + a_{23}x_3(0)\right) - k_{14}\left(a_{42}x_2(0) + a_{43}x_3(0)\right)\end{aligned}$$

$$\begin{aligned}\dot{k}_2 = {} & 2b_4^2k_4k_{24} - \frac{1}{2}(q_{12}x_{1d} + q_{22}x_{2d} + q_{23}x_{3d} + q_{24}x_{4d}) \\ & -k_{22}\left(a_{22}x_2(0) + a_{23}x_3(0)\right) - k_{24}\left(a_{42}x_2(0) + a_{43}x_3(0)\right) \\ & - k_1 - a_{22}k_2 - a_{42}k_4\end{aligned}$$

$$\dot{k}_3 = 2b_4^2 k_4 k_{34} - \frac{1}{2}(q_{13}x_{1d} + q_{23}x_{2d} + q_{33}x_{3d} + q_{34}x_{4d})$$

$$- k_{23}\big(a_{22}x_2(0) + a_{23}x_3(0)\big) - k_{34}\big(a_{42}x_2(0) + a_{43}x_3(0)\big)$$

$$- a_{23}k_2 - a_{43}k_4$$

$$\dot{k}_4 = 2b_4^2 k_4 k_{44} - \frac{1}{2}(q_{14}x_{1d} + q_{24}x_{2d} + q_{34}x_{3d} + q_{44}x_{4d})$$

$$- k_{24}\big(a_{22}x_2(0) + a_{23}x_3(0)\big) - k_{44}\big(a_{42}x_2(0) + a_{43}x_3(0)\big)$$

$$- k_3 - a_{44}k_4$$

$$\dot{k}_{11} = 2b_4^2 k_{14}^2 - \frac{1}{2} q_{11}$$

$$\dot{k}_{22} = 2b_4^2 k_{24}^2 - \frac{1}{2} q_{22} - 2k_{12} - 2a_{22}k_{22} - 2a_{42}k_{24}$$

$$\dot{k}_{33} = 2b_4^2 k_{34}^2 - \frac{1}{2} q_{33} - 2a_{23}k_{23} - 2a_{43}k_{34}$$

$$\dot{k}_{44} = 2b_4^2 k_{44}^2 - \frac{1}{2} q_{44} - 2k_{34} - 2a_{44}k_{44}$$

$$\dot{k}_{12} = 2b_4^2 k_{14}k_{24} - \frac{1}{2} q_{12} - k_{11} - a_{22}k_{12} - a_{42}k_{14}$$

$$\dot{k}_{13} = 2b_4^2 k_{14}k_{34} - \frac{1}{2} q_{13} - a_{23}k_{12} - a_{43}k_{14}$$

$$\dot{k}_{14} = 2b_4^2 k_{14}k_{44} - \frac{1}{2} q_{14} - k_{13} - a_{44}k_{14}$$

$$\dot{k}_{23} = 2b_4^2 k_{24}k_{34} - \frac{1}{2} q_{23} - k_{13} - a_{22}k_{23} - a_{23}k_{22} - a_{42}k_{34}$$

$$- a_{43}k_{24}$$

$$\dot{k}_{24} = 2b_4^2 k_{24}k_{44} - \frac{1}{2} q_{24} - k_{14} - k_{23} - a_{22}k_{24} - a_{42}k_{44} - a_{44}k_{24}$$

$$\dot{k}_{34} = 2b_4^2 k_{34}k_{44} - \frac{1}{2} q_{34} - k_{33} - a_{23}k_{24} - a_{43}k_{44} - a_{44}k_{34}$$

where the boundary conditions at $t = t_f$ must be specified.

Equation (7.3) is the expression for E at $t = t_f$. Equating this to (7.12) evaluated at $t = t_f$ gives

$$(B.6)\quad k(t_f) - 2\sum_{m=1}^{4} k_m(t_f)\, x_m(t_f) + \sum_{m=1}^{4}\sum_{p=1}^{4} k_{mp}(t_f)x_m(t_f)x_p(t_f)$$

$$= \frac{1}{2}\left\{ [\underline{x}(t_f) - \underline{x}_d(t_f)]^T F\, [\underline{x}(t_f) - \underline{x}_d(t_f)]\right\}$$

Expanding both sides and solving for the $k(t_f)$'s gives the boundary conditions on the k's:

$$(B.7)\quad k(t_f) = \frac{1}{2}[f_{11}x_{1d}^2(t_f) + f_{22}x_{2d}^2(t_f) + f_{33}x_{3d}^2(t_f) + f_{44}x_{4d}^2(t_f)] + f_{12}x_{1d}(t_f)x_{2d}(t_f) + f_{13}x_{1d}(t_f)x_{3d}(t_f) + f_{14}x_{1d}(t_f)x_{4d}(t_f) + f_{23}x_{2d}(t_f)x_{3d}(t_f) + f_{24}x_{2d}(t_f)x_{4d}(t_f) + f_{34}x_{3d}(t_f)x_{4d}(t_f)$$

$$k_1(t_f) = \frac{1}{2}[f_{11}x_{1d}(t_f) + f_{12}x_{2d}(t_f) + f_{13}x_{3d}(t_f) + f_{14}x_{4d}(t_f)]$$

$$k_2(t_f) = \frac{1}{2}[f_{12}x_{1d}(t_f) + f_{22}x_{2d}(t_f) + f_{23}x_{3d}(t_f) + f_{24}x_{4d}(t_f)]$$

$$k_3(t_f) = \frac{1}{2}[f_{13}x_{1d}(t_f) + f_{23}x_{2d}(t_f) + f_{33}x_{3d}(t_f) + f_{34}x_{4d}(t_f)]$$

$$k_4(t_f) = \frac{1}{2}[f_{14}x_{1d}(t_f) + f_{24}x_{2d}(t_f) + f_{34}x_{3d}(t_f) + f_{44}x_{4d}(t_f)]$$

$$k_{11}(t_f) = \frac{1}{2} f_{11}$$

$$k_{22}(t_f) = \frac{1}{2} f_{22}$$

$$k_{33}(t_f) = \frac{1}{2} f_{33}$$

$$k_{44}(t_f) = \frac{1}{2} f_{44}$$

$$k_{12}(t_f) = \frac{1}{2} f_{12}$$

$$k_{13}(t_f) = \frac{1}{2} f_{13}$$

$$k_{14}(t_f) = \frac{1}{2} f_{14}$$

$$k_{23}(t_f) = \frac{1}{2} f_{23}$$

$$k_{24}(t_f) = \frac{1}{2} f_{24}$$

$$k_{34}(t_f) = \frac{1}{2} f_{34}$$

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13. ABSTRACT

An automatic elevator controller for the final phase of an Instrument Landing System approach is designed using optimization theory and the practicality of the controller investigated. The problem is discussed and the assumptions stated. Then a mathematical model for the aircraft and a desired flare-out approach path are derived. The aircraft and approach limitations are established and the model is tested. Dynamic Programming and the Parametric Expansion Method provide the optimal control from which the controller is designed. A computer program is developed to investigate the controller. The results are inconclusive and a recommendation for further study is made.

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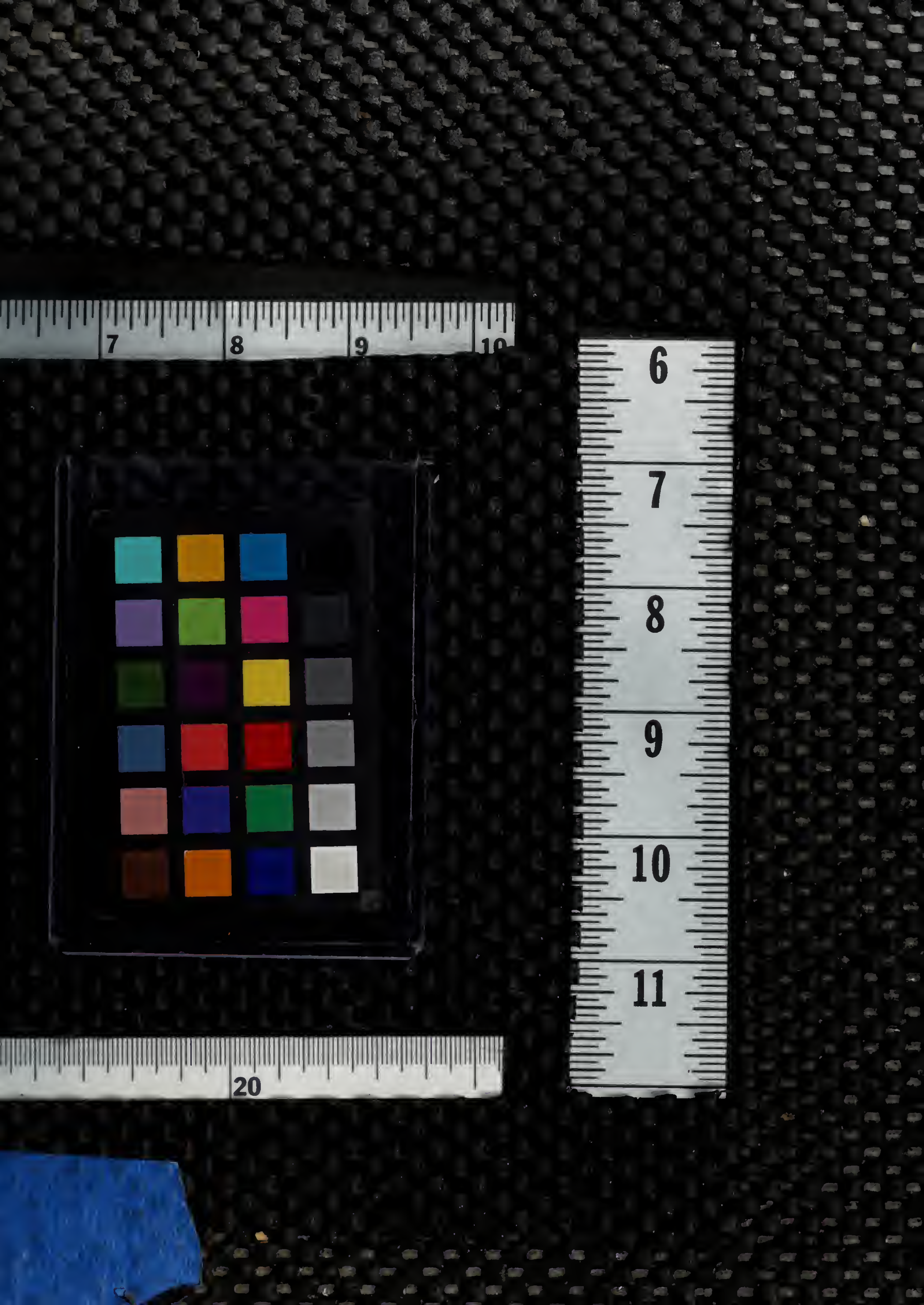

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